The inverse eigenvalue problem of a graph: Multiplicities and minors

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Abstract

The inverse eigenvalue problem of a given graph G is to determine all possible spectra of real symmetric matrices whose off-diagonal entries are governed by the adjacencies in G. Barrett et al. introduced the Strong Spectral Property (SSP) and the Strong Multiplicity Property (SMP) in [8]. In that paper it was shown that if a graph has a matrix with the SSP (or the SMP) then a supergraph has a matrix with the same spectrum (or ordered multiplicity list) augmented with simple eigenvalues if necessary, that is, subgraph monotonicity. In this paper we extend this to a form of minor monotonicity, with restrictions on where the new eigenvalues appear. These ideas are applied to solve the inverse eigenvalue problem for all graphs of order five, and to characterize forbidden minors of graphs having at most one multiple eigenvalue.

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1 Introduction

Inverse eigenvalue problems appear in various contexts throughout mathematics and engineering, and refer to determining all possible lists of eigenvalues (spectra) for matrices fitting some description. Graphs often describe relationships in a physical setting, such as control of a system, and the eigenvalues of associated matrices govern the behavior of the system. The *inverse eigenvalue problem of a graph (IEPG)* refers to determining the possible spectra of real symmetric matrices whose pattern of nonzero off-diagonal entries is described by the edges of a given graph. The IEPG is motivated by inverse problems arising in the theory of vibrations. The study of the vibrations of a string leads to classical inverse Sturm–Liouville problems, and its generalizations continue

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to be an active area of research (see [16] and references therein). The IEPG where G is a path corresponds to a discretization of the inverse Sturm–Liouville problem for the string, and leads to the classical study of the inverse eigenvalue problem for Jacobi matrices (that is, irreducible, tridiagonal matrices) that was resolved by the sequence of papers by Downing and Householder [12], and Hochstadt [17, 18]. Thus, as noted in [16], the IEPG can be viewed as the inverse problem for a vibrating system with prescribed structure given by G.

The IEPG was originally approached through the study of ordered multiplicity lists for eigenvalues. It was thought by many researchers in the field that at least for a tree T, determining the ordered multiplicity lists of T would suffice to determine the spectra of matrices described by T. When it was shown in [4] that this was not the case, the focus of much of the research in the area shifted to the narrower question of maximum eigenvalue multiplicity, or equivalently maximum nullity or minimum rank of matrices described by the graph. While the maximum multiplicity has been determined for many families of graphs, including all trees, in general it remains an open question and active area of research (see [13, 14] for extensive bibliographies). More recently, there has been progress on the related question of determining the minimum number of distinct eigenvalues of matrices described by a given graph [1, 8].

Maximum nullity, minimum number of distinct eigenvalues, and ordered multiplicity lists all provide information that can in some cases be used to solve the inverse eigenvalue problem for a specific graph or family of graphs, but the question of the structures or properties that are necessary to allow this to be done more generally is open. Recently new tools, the Strong Spectral Property (SSP) and the Strong Multiplicity Property (SMP), were developed [8]; these offer hope for progress on what was previously seen as an intractable problem.

In Section 3 we use the SSP to solve the IEPG for graphs of order 5 (order 4 was solved in [9], but the SSP provides a shorter proof). Every graph of order 5 that allows an SMP matrix for an ordered multiplicity list also allows an SSP matrix for the same ordered multiplicity list. This naturally raises the question of whether there exists a graph that has a matrix $A \in \mathcal{S}(G)$ with the SMP, but does not allow the SSP for any matrix having the ordered multiplicity list of A. In Section 4 we exhibit such a matrix and establish its properties. (Of course it is much easier to construct a matrix that has the SMP but not the SSP; see, for example, [8, Remark 22].)

Another main result of this paper, which we use to establish characterizations of multiple eigenvalues by forbidden minors, is the Minor Monotonicity Theorem (Theorem 6.13). This theorem says that if G is a minor of H and $A \in \mathcal{S}(G)$ has the SSP (or the SMP), then there is a matrix $B \in \mathcal{S}(H)$ with the SSP (or the SMP) and $\operatorname{spec}(A) \subseteq \operatorname{spec}(B)$ (or the ordered multiplicity list of B can be obtained from that of A by adding **ones**). Additional eigenvalues are added as simple #1 eigenvalues and can always be added at the ends of the spectrum of A; in the case of a vertex deletion they can be added arbitrarily, but for a minor obtained by contraction the new eigenvalues may be restricted to being sufficiently far from the spectrum of A.

Minor monotonicity enables forbidden minor characterizations, and as an illustration of this in Section 5 we use Theorem 6.13 to determine the forbidden minors for a graph to have at most one multiple eigenvalue and to not have two consecutive multiple eigenvalues.

In Section 7 we establish additional tools for constructing matrices with prescribed spectra and multiplicity lists. The foundation is the Matrix Liberation Lemma (Lemma 7.3), which describes how the three strong properties indicate the ability of a matrix to effect any sufficiently small perturbation of its nonzero entries with complete freedom while preserving its rank (SAP), its exact spectrum (SSP), or its ordered list of eigenvalue multiplicities (SMP). One consequence of this is the Augmentation Lemma (Lemma 7.5), which gives conditions under which an eigenvalue λ of an SSP matrix $A \in \mathcal{S}(G-v)$ guarantees the existence of a matrix $B \in \mathcal{S}(G)$ with $\operatorname{spec}(B) = \operatorname{spec}(A) \cup \{\lambda\}$ (i.e., $\operatorname{mult}_B(\lambda) = \operatorname{mult}_A(\lambda) + 1$ and the other eigenvalues and multiplicities are unchanged from A to B).

In the next section we introduce the necessary terminology, including definitions of the SSP and the SMP. Sections 6 and 7 are rather technical and do not use any results from Section 3, 4, and 5, so we defer the proofs of the results therein to the later part of the paper.

2 Terminology, notation, and background

All matrices are real and symmetric; O and I denote zero and identity matrices of appropriate size, respectively. If the distinct eigenvalues of A are $\lambda_1 < \lambda_2 < \cdots < \lambda_q$ and the multiplicities of these eigenvalues are $m_1, m_2, \ldots, m_q$ respectively, then the *ordered multiplicity list of A* is $\mathbf{m}(A) = (m_1, m_2, \ldots, m_q)$. The *spectral radius* of A is $\rho(A) = \max\{|\lambda| : \lambda \in \operatorname{spec}(A)\}$. For an $n \times n$ matrix M and $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$, the submatrix of M lying in rows indexed by α and columns indexed by β is denoted by $M[\alpha, \beta]$; in the case that $\beta = \{1, 2, \ldots, n\}$ this can be denoted by $M[\alpha, :]$, and similarly for $\alpha = \{1, 2, \ldots, n\}$. For brevity, $M[\alpha]$ means $M[\alpha, \alpha]$. Let $\overline{\alpha} = \{1, 2, \ldots, n\} \setminus \alpha$ and $\overline{\beta}$ defined similarly. Then $M(\alpha, \beta) = M[\overline{\alpha}, \overline{\beta}]$ and $M(\alpha) = M[\overline{\alpha}]$. #10 (1006)

A symmetric matrix A has the *Strong Arnold Property* (or A has the SAP for short) if the only symmetric matrix X satisfying $A \circ X = O$, $I \circ X = O$ and $AX = O$ is $X = O$. An $n \times n$ symmetric matrix A satisfies the *Strong Multiplicity Property* (or A has the SMP) provided the only symmetric matrix X satisfying $A \circ X = O$, $I \circ X = O$, $[A, X] = O$, and $\operatorname{tr}(A^i X) = 0$ for $i = 2, \ldots, n-1$ is $X = O$ [8, Definition 18 and Remark 19]. A symmetric matrix A has the *Strong Spectral Property* (or A has the SSP) if the only symmetric matrix X satisfying $A \circ X = O$, $I \circ X = O$ and $[A, X] = O$ is $X = O$ [8, Definition 8]. Clearly the SSP implies the SMP, and the SMP implies $A + \lambda I$ has the SAP for every real number λ.

The *graph* $\mathcal{G}(A)$ of a symmetric $n \times n$ matrix A is the (simple, undirected, finite) graph with vertices $\{1, \ldots, n\}$ and edge ij such that $i \neq j$ and $a_{ij} \neq 0$. For a graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge set E, the *set of symmetric matrices described by G*, $\mathcal{S}(G)$, is the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that $\mathcal{G}(A) = G$. The IEPG for G asks for the determination of all possible spectra of matrices in $\mathcal{S}(G)$. The *maximum multiplicity* of G is $\operatorname{M}(G) = \max\{\operatorname{mult}_A(\lambda) : A \in \mathcal{S}(G),\ \lambda \in \operatorname{spec}(A)\}$, and the *minimum rank* of G is $\operatorname{mr}(G) = \min\{\operatorname{rank} A : A \in \mathcal{S}(G)\}$. It is easily seen that $\operatorname{M}(G) = \max\{\operatorname{null} A : A \in \mathcal{S}(G)\}$, so $\operatorname{M}(G)$ is also called the *maximum nullity* of G, and $\operatorname{mr}(G) + \operatorname{M}(G) = |G|$, where $|G|$ is the number of vertices of G. The number of distinct eigenvalues of A is denoted by $q(A)$, and $q(G) = \min\{q(A) : A \in \mathcal{S}(G)\}$.

If v is a vertex of a graph G, the *neighborhood* of n is the set of vertices adjacent to v, and is denoted by $N_G(v)$. The *closed neighborhood* of v is $N_G[v] := N_G(v) \cup \{v\}$. The *complement* $\overline{G}$ of G is the graph with the same vertex set as G and edges exactly where G does not have edges. A *generalized star* is a tree with at most one vertex of degree three or more. A *unicyclic graph* is a graph with one cycle.

The well known Parter–Wiener Theorem for trees plays a fundamental role in the study of the IEPG, and we state it here.

Theorem 2.1 (Parter–Wiener Theorem)**.** [21, 24, 26] *Let T be a tree, $A \in \mathcal{S}(T)$ and $\operatorname{mult}_A(\lambda) \geq 2$. Then there exists a vertex v such that $\operatorname{mult}_{A(v)}(\lambda) = \operatorname{mult}_A(\lambda) + 1$ and λ is an eigenvalue of the principal submatrices of A corresponding to at least three components of $T - v$.*

There are several (well known) consequences of Theorem 2.1 and interlacing (see, for example, [20]).

Lemma 2.2. *The first and last eigenvalue of a tree must be simple.*

Lemma 2.3. *If T is a generalized star and $A \in \mathcal{S}(T)$, then A cannot have consecutive multiple eigenvalues.*

3 Inverse eigenvalue problem for graphs of order at most five

In this section we solve the IEP for all graphs of order at most five, both in general and with the stipulation that there is a matrix realizing each possible spectrum that has the SSP (or equivalently, the SMP). The solution to the IEP for graphs of order at most three is well known and straightforward, and the IEP for graphs of order 4 was solved in [9]. In Section 3.1 we briefly re-derive the solution for order 4 by using the SSP to shorten the arguments. We then solve the IEP for graphs of order 5 in Section 3.2.

An ordered multiplicity list $\mathbf{m} = (m_1, \ldots, m_k)$ is *spectrally arbitrary* for graph G if any set of k real numbers $\lambda_1 < \cdots < \lambda_k$ can be realized as the spectrum of $A \in \mathcal{S}(G)$ with $\operatorname{mult}_A(\lambda_i) = m_i$ (and $\mathbf{m}$ is spectrally arbitrary for G with the SSP has the obvious interpretation). Figure 1 summarizes the solution for connected graphs of order at most 5, both with the SSP and without the SSP. Note that some graph names are abbreviated: Banner (Bnr), Butterfly (Bfly), Diamond (Dmnd), Full House (FHs), House (Hs). We show that for a connected graph of order at most 4, every ordered multiplicity list that is attainable by the graph is spectrally arbitrary with the SSP. For order 5 every ordered multiplicity list that is attainable (respectively, attainable with the SSP) is spectrally arbitrary (spectrally arbitrary with the SSP), but there are connected graphs and ordered multiplicity lists that can be realized only without the SSP. In all cases disconnected graphs have some ordered multiplicity lists that can be realized only without the SSP. While it is not true that all trees are spectrally arbitrary [4], we show that it is true for all graphs of order at most five.

Remark 3.1. Spectra for disconnected graphs can be determined from those of connected graphs by allowing any permissible assignment of disjoint spectra to the connected components when SSP is required [8, Theorem 34], and when SSP is not required any permissible assignment of spectra to the connected components. Thus the solutions to the inverse eigenvalue problem with and without SSP are different.

Remark 3.2. Given a matrix A that has exactly two distinct eigenvalues $\lambda_1 < \lambda_2$ with multiplicities m_1 and m_2, and any possible pair of real numbers $\mu_1 < \mu_2$, the matrix

$$B = \frac{\mu_2 - \mu_1}{\lambda_2 - \lambda_1}(A - \lambda_1 I) + \mu_1 I$$

has eigenvalues μ_1 with multiplicity m_1 and μ_2 with multiplicity m_2. This technique is referred to as *scale and shift*. By negation of the matrix, the order of multiplicities can be reversed. Negation and scale and shift all preserve the SSP, meaning that if A has the SSP, then so does $\alpha A + \beta I$ for any nonzero real number α and any $\beta \in \mathbb{R}$. Thus for an ordered multiplicity list with only two multiplicities, exhibiting one matrix (respectively, one matrix with the SSP) with this ordered multiplicity list suffices to show the graph is spectrally arbitrary (respectively, spectrally arbitrary with the SSP) for this ordered multiply list.

For every graph of order n, any set of n distinct real numbers is attained by a matrix with the SSP [8, Remark 15]. Thus $(1, 1, \ldots, 1)$ is spectrally arbitrary with SSP for every graph. Only distinct eigenvalues are possible for a path. This covers all connected graphs of order at most 3

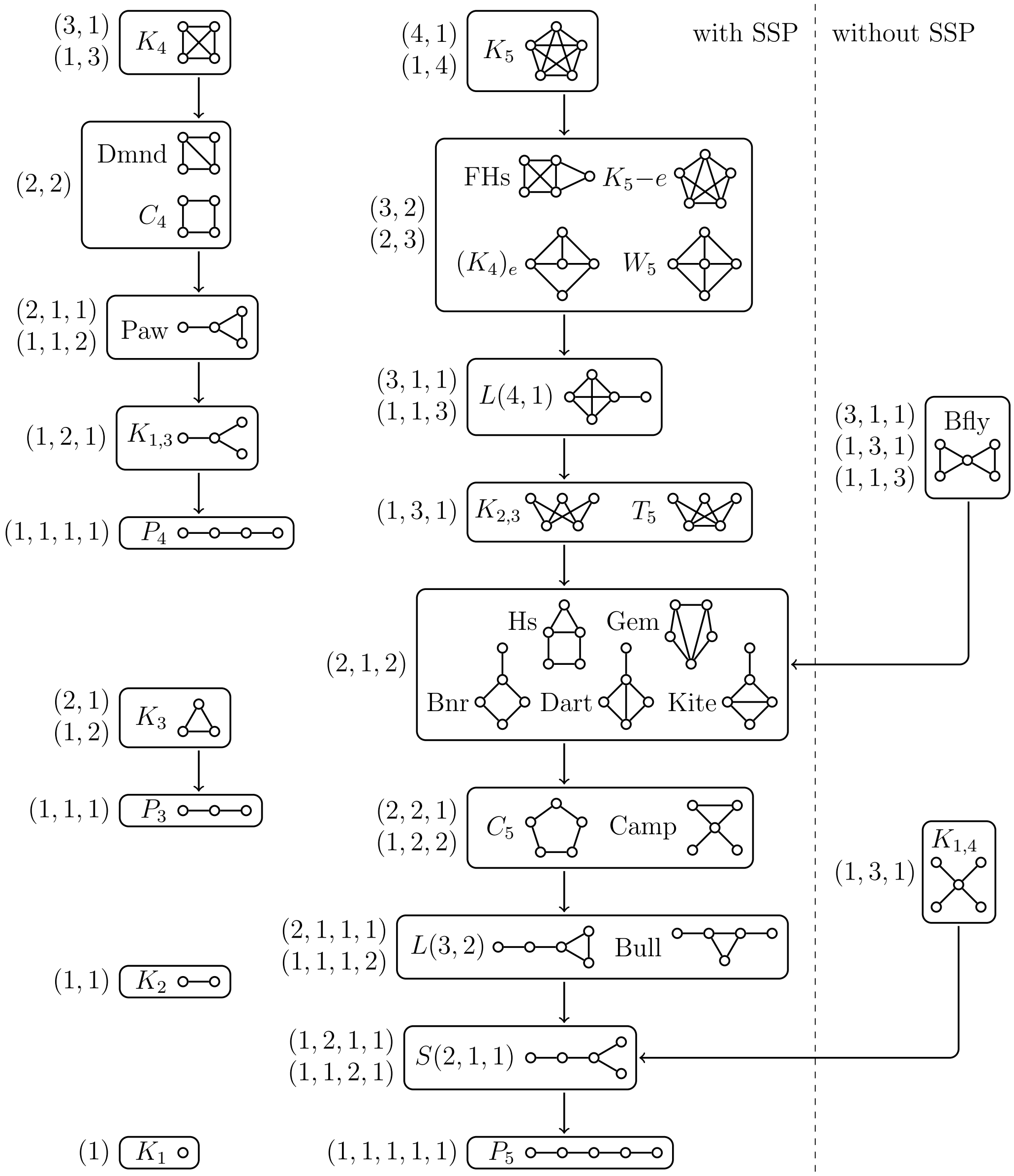

Figure 1: The connected graphs of order at most 5 with their ordered multiplicity lists. If a box is joined to another box by a line then the graphs in the upper box can realize every ordered multiplicity list of the graphs in the lower box (including other boxes below connected with lines to lower boxes). Every ordered multiplicity list is spectrally arbitrary for the graphs that attain it.

except K_3. Any $A \in \mathcal{S}(K_3)$ has the SSP, and J_3 has ordered multiplicity list $(2, 1)$. The list (3) cannot be attained by a connected graph.

If G is a subgraph of H on the same vertex set, then any spectrum attained by G with SSP is also attained by H with SSP [8, Theorem 10]. Thus it is useful to identify minimal graphs attaining a given ordered multiplicity list and show that these are spectrally arbitrary. These minimal subgraphs need not be connected.

We state an additional result that will be used.

Lemma 3.3. *Suppose G is a connected unicyclic graph with an odd cycle. At least one of the first and last eigenvalues of G must be simple.*

Proof. Let $A \in \mathcal{S}(G)$. If the cycle product (product of the entries of A on the cycle) is negative, then replace A by $-A$; since the cycle is odd, the cycle product is now positive. Let G' be obtained from G by deleting one edge $\{i, j\}$ of the cycle, so G' is a spanning tree of G. Let A' be defined from A by replacing (the equal) entries a_{ij} and a_{ji} by 0, so $A' \in \mathcal{S}(G')$. There exists a diagonal matrix D with diagonal entries ± 1 so that $DA'D = DA'D^{-1}$ is a nonnegative matrix [11, Lemma 1.2]. Since the cycle product of A is unchanged by a diagonal similarity, the cycle product of DAD is positive, implying DAD is nonnegative. Since DAD is symmetric, it has 2-cycles and it has an odd cycle, so DAD is primitive. The spectral radius of a primitive nonnegative matrix is simple, so the last eigenvalue of A is simple. $\square$

3.1 Order 4

The next result describes the solution to the IEP for connected graphs of order 4; as noted in Remark 3.1, the solution for orders 1, 2, and 3 suffice to solve the IEP for disconnected graphs of order 4.

Proposition 3.4.

1. *Table 1 lists all the minimal subgraphs with respect to the SSP for each ordered multiplicity list of order 4. In each case the reversal of the given list has the same minimal subgraphs.*

2. *The order 4 part of Figure 1 lists all the possible ordered multiplicity lists for each connected graph of order 4 (both those ordered multiplicity lists next to the box with the graph and all those next to graphs below and connected by a sequence of lines in the order 4 diagram). Each ordered multiplicity list in the order 4 part of Figure 1 is spectrally arbitrary with the SSP.*

Table 1: Minimal order 4 subgraphs for attainment with SSP, by ordered multiplicity list (OML)

OML	graph	reason	graph	reason	graph	reason
(3,1)	K_4	J_4				
(2,2)	C_4	$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$				
(1,2,1)	C_4	Corollary 7.6	$K_3 \dot\cup K_1$	$\left(\frac{\lambda_3-\lambda_2}{3} J_3 + \lambda_2 I_3\right) \oplus [\lambda_1]$	$K_{1,3}$	$\begin{pmatrix} a & b & b & b \\ b & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ b & 0 & 0 & 0 \end{pmatrix}$
(2,1,1)	C_4	Corollary 7.6	$K_3 \dot\cup K_1$	$\left(\frac{\lambda_2-\lambda_1}{3} J_3 + \lambda_1 I_3\right) \oplus [\lambda_3]$		
(1,1,1,1)	$4K_1$	[8, Remark 15]				

Proof. In each case a matrix or reason is listed in Table 1. It is straightforward to verify that each matrix has the SSP regardless of the parameters, and to see that the matrix for $K_{1,3}$ for (1,2,1) is spectrally arbitrary by choice of parameters (and scale and shift); see [23] for verifications of the SSP. Thus each ordered multiplicity list in Table 1 is spectrally arbitrary with the SSP for the listed graph.

It is then straightforward to verify that every ordered multiplicity list in the order 4 part of Figure 1 is spectrally arbitrary with SSP for the graphs in its box or above it, by use of a subgraph with the ordered multiplicity list that is spectrally arbitrary with SSP (cf. Table 1).

To complete the proof, we justify that no other ordered multiplicity list is possible for each graph, which also justifies that all minimal subgraphs are listed in Table 1:

- For K_4, $\mathrm{M}(K_4) = 3$.
- For C_4 and Diamond, $\mathrm{M}(C_4) = 2 = \mathrm{M}(\mathsf{Dmnd})$.
- Paw has at least 3 distinct eigenvalues ($q(\text{Paw}) = 3$) by [1, Theorem 3.2] since there is a unique path of length 2 from the degree one vertex to either degree two vertex.
- For $K_{1,3}$, the first and last eigenvalues are simple (Lemma 2.2) and $\mathrm{M}(K_{1,3}) = 2$.
- For P_4, the maximum multiplicity is $\mathrm{M}(P_4) = 1$. $\square$

3.2 Order 5

Lemma 3.5. *Let*

$$M_1 = \begin{pmatrix} -t^4+2t^3-t^2-1 & 0 & -(t-1)t & (t-1)^2t^2 & 0 \\ 0 & -t^4+2t^3-t^2-1 & 0 & -(t-1)t^2 & -(t-1)t \\ -(t-1)t & 0 & -t^2\left(t^2-2t+2\right) & 0 & -(t-1)t^2 \\ (t-1)^2t^2 & -(t-1)t^2 & 0 & -t^2\left(t^2-2t+2\right) & 0 \\ 0 & -(t-1)t & -(t-1)t^2 & 0 & -2(t-1)^2t^2 \end{pmatrix}, 0 < t < 1,$$

$$M_2 = \begin{pmatrix} -1 & 1 & -a & 0 & 0 \\ 1 & -1 & -a & 0 & 0 \\ -a & -a & 2a^2-2 & -a & -a \\ 0 & 0 & -a & 0 & 0 \\ 0 & 0 & -a & 0 & 0 \end{pmatrix}, a \neq 0, \qquad M_3 = \begin{pmatrix} 1 & -1 & 0 & 0 & -a \\ -1 & 1 & 0 & 0 & -a \\ 0 & 0 & -a^2 & a^2 & a \\ 0 & 0 & a^2 & -a^2 & a \\ -a & -a & a & a & 2-2a^2 \end{pmatrix}, a \neq 0,$$

$$M_4 = \begin{pmatrix} a & 0 & b & b & b \\ 0 & -ac^2 & bc & bc & bc \\ b & bc & 0 & 0 & 0 \\ b & bc & 0 & 0 & 0 \\ b & bc & 0 & 0 & 0 \end{pmatrix}, b \neq 0, c \neq 0, \pm 1, \qquad M_5 = \begin{pmatrix} a^2 & 0 & \sqrt{2}a & a & a \\ 0 & 1 & -\sqrt{2} & 1 & 1 \\ \sqrt{2}a & -\sqrt{2} & 4 & 0 & 0 \\ a & 1 & 0 & 2 & 2 \\ a & 1 & 0 & 2 & 2 \end{pmatrix}, a \geq 1.$$

(1) $M_i, i = 1, 2, 3, 4, 5$ has the SSP for any permitted parameters.

(2) $\mathcal{G}(M_1) = C_5$, $\mathcal{G}(M_2) = \mathsf{Camp}$, $\mathcal{G}(M_3) = \mathsf{Bfly}$, $\mathcal{G}(M_4) = K_{2,3}$, and $\mathcal{G}(M_5) = (K_4)_e$.

(3) $\mathbf{m}(M_1) = (2, 2, 1)$, and $\lambda_1 < \lambda_2 < \lambda_3$ can be realized with M_1 by choosing t and scale and shift.

(4) $\mathbf{m}(M_2) = (2, 2, 1)$, and $\lambda_1 < \lambda_2 < \lambda_3$ can be realized with M_2 by choosing a and scale and shift.

(5) $\mathbf{m}(M_3) = (2, 1, 2)$, and $\lambda_1 < \lambda_2 < \lambda_3$ can be realized with M_3 by choosing a and scale and shift.

(6) $\mathbf{m}(M_4) = (1, 3, 1)$, and $\lambda_1 < \lambda_2 < \lambda_3$ can be realized with M_4 by choosing a, b, c and scale and shift.

(7) $\mathbf{m}(M_5) = (3, 1, 1)$ for $a > 1$, and $\lambda_1 < \lambda_2 < \lambda_3$ can be realized by M_5 by choosing $a > 1$ and scale and shift. $\mathbf{m}(M_5) = (3, 2)$ for $a = 1$, and $\lambda_1 < \lambda_2$ can be realized by M_5 with $a = 1$ by scale and shift.

Proof. In each case it is clear that M_i has the stated graph, and it is straightforward to verify the SSP for each matrix (see [23]).

For statements (4), (5), and (7), examination of the eigenvalues shows that the claimed ordered multiplicity list is realized, and any spectrum can be realized by appropriate choice of the parameter and scale and shift; here we list the spectra:

$$\begin{aligned}\operatorname{spec}(M_2) &= \{-2,-2,0,0,2a^2\}\\ \operatorname{spec}(M_3) &= \{-2a^2,-2a^2,0,2,2\}\\ \operatorname{spec}(M_5) &= \{0,0,0,5,4+a^2\}\end{aligned}$$

Computations show that $\operatorname{spec}(M_1) = \{\lambda,\lambda,\mu,\mu,0\}$ with

$$\begin{aligned}\lambda &= \frac{1}{2}\left(-3t^4+6t^3-4t^2-1-(1-t)\sqrt{t^6-2t^5+3t^4+3t^2+2t+1}\right) \text{ and}\\ \mu &= \frac{1}{2}\left(-3t^4+6t^3-4t^2-1+(1-t)\sqrt{t^6-2t^5+3t^4+3t^2+2t+1}\right).\end{aligned}$$

Furthermore, $\lambda < \mu < 0$ where the last inequality follows from $\lambda < 0$ and Lemma 3.3. Note that #2 λ and μ are continuous functions of t with $\lim_{t\to 0}\frac{\mu}{\lambda} = 0$ and $\lim_{t\to 1}\frac{\mu}{\lambda} = 1$. So given $\alpha_1 < \alpha_2 < 0$, we can choose t so that $\frac{\mu}{\lambda} = \frac{\alpha_2}{\alpha_1}$. Then scaling M_1 achieves eigenvalues $\{\alpha_1,\alpha_1,\alpha_2,\alpha_2,0\}$, and finally a shift is made if needed.

For the matrix M_4 with ordered multiplicity list (1,3,1), the characteristic polynomial is

$$p_{M_4}(x) = x^3\left(-a^2c^2-3b^2(1+c^2)+a(-1+c^2)x+x^2\right).$$

Any eigenvalues λ and μ of opposite sign can be realized by

$$a = \frac{-\lambda-\mu}{c^2-1},\ b = \frac{\sqrt{-c^4\lambda\mu-c^2\lambda^2-c^2\mu^2-\lambda\mu}}{\sqrt{3}\sqrt{c^6-c^4-c^2+1}}.$$

The only restrictions on the parameters are $b \neq 0, c \neq 0, c \neq \pm 1$, $-c^4\lambda\mu - c^2\lambda^2 - c^2\mu^2 - \lambda\mu > 0$, and $c^6 - c^4 - c^2 + 1 > 0$. Since $-\lambda\mu > 0$, all the inequalities can be ensured by choosing c sufficiently small. □

Theorem 3.6.

1. *Table 2 lists all the minimal subgraphs with respect to the SSP for each ordered multiplicity list of order* 5*. In each case the reversal of the given list has the same minimal subgraphs.*

2. *The order 5 part of Figure 1 lists all the possible ordered multiplicity lists for each connected graph of order 5 (both those ordered multiplicity lists next to the box with the graph and all those next to graphs below and connected by a sequence of lines in the order 5 diagram). Each ordered multiplicity list in Figure 1 is spectrally arbitrary. Those to the left of the dashed vertical line can realized with the SSP, whereas those to the right cannot be realized with the SMP (and thus not with the SSP).*

Proof. For each connected graph in Table 2, a matrix or reason is listed. That the matrices listed have the given ordered multiplicity list and can realize any such spectra was established in Lemma 3.5, and it is straightforward to verify that each of the listed matrices has the SSP (see [23]). For the disconnected graphs, the result follows from Proposition 3.4 and the Block Diagonal Theorem [8, Theorem 34].

Table 2: Minimal order 5 subgraphs for attainment with SSP, by ordered multiplicity list (OML)

OML	graph	reason	graph	reason	graph	reason	graph	reason
(4,1)	K_5	J_5						
(3,2)	$(K_4)_e$	$M_5, a = 1$						
(3,1,1)	$K_4 \dot\cup K_1$		$(K_4)_e$	$M_5, a > 1$				
(1,3,1)	$K_4 \dot\cup K_1$		$K_{2,3}$	M_4				
(2,1,2)	$C_4 \dot\cup K_1$		Butterfly	M_3				
(2,2,1)	$C_4 \dot\cup K_1$		C_5	M_1	Campstool	M_2		
(2,1,1,1)	$C_4 \dot\cup K_1$		C_5	Corollary 7.6	$K_3 \dot\cup 2K_1$			
(1,2,1,1)	$C_4 \dot\cup K_1$		C_5	Corollary 7.6	$K_3 \dot\cup 2K_1$		$K_{1,3} \dot\cup K_1$	
(1,1,1,1,1)	$5K_1$							

The information in Table 2 can be used to justify the following statement: Every ordered multiplicity list to the left of the dashed vertical line in the order 5 part of Figure 1 is spectrally arbitrary with the SSP for the graphs in its box or above it. In some cases a subgraph with the desired ordered multiplicity list that is spectrally arbitrary with the SSP is used for the justification.

To complete the proof, we show that for each graph no ordered multiplicity list other than those described in Figure 1 is possible, which also justifies that all minimal subgraphs are listed in Table 2, and discuss why the non-SSP ordered multiplicity lists cannot be realized with SMP. In each case below, the statements about M (maximum multiplicity = maximum nullity), M_+ (maximum positive semidefinite nullity) and ξ (maximum nullity with SAP) are well known (see, for example, [2], [7], and [14]).

- For P_5, $\mathrm{M}(P_5) = 1$.
- For $G = S(2,1,1)$, the first and last eigenvalues are simple since G is a tree (Lemma 2.2), and $\mathrm{M}(G) = 2$.
- For $K_{1,4}$: The first and last eigenvalues are simple, so the only possible ordered multiplicity list that has not already been shown to be realized with the SSP is $(1,3,1)$. The adjacency matrix realizes $(1,3,1)$, and it is known that any star is spectrally arbitrary for every ordered multiplicity list it attains [4]. Since $\xi(K_{1,4}) = 2$ and the SMP implies the SAP, $(1,3,1)$ cannot be realized with the SMP.
- For $G = L(3,2)$ or Bull, $\mathrm{M}(G) = 2$, $q(G) = 4$, and one of the first and last eigenvalues is simple by Lemma 3.3.
- For $G = C_5$ or Campstool (Camp), $\mathrm{M}(G) = 2$ and one of the first and last eigenvalues is simple by Lemma 3.3.
- For G one of the graphs Banner (Bnr), House (Hs), Dart, Gem, or Kite, $\mathrm{M}(G) = 2$.
- For $G =$ Butterfly (Bfly): $\mathrm{M}(G) = 3$ and $\xi(G) = 2$, so no ordered multiplicity list containing a 3 can be realized with the SMP. Without the SMP, it is known that (1,3,1) and (3,1,1) are spectrally arbitrary for Butterfly [22, Section 5.2].
- For $G = K_{2,3}$ or T_5, $\mathrm{M}(G) = 3$ and $\mathrm{M}_+(G) = 2$; the latter eliminates $(3,1,1)$ and $(3,2)$ as possible ordered multiplicity lists.
- For $L(4,1)$, $\mathrm{M}(L(4,1)) = 3$ and $q(L(4,1)) = 3$ by unique shortest path of length 2.
- For G one of Full House (FHs), $K_5 - e$, $(K_4)_e$, or W_5, $\mathrm{M}(G) = 3$.
- For K_5, $\mathrm{M}(K_5) = 4$. □

4 A graph and ordered multiplicity list that allow the SMP but not the SSP

In this section we exhibit a graph G and ordered multiplicity list $\mathbf{m}$ such if $A \in \mathcal{S}(G)$ and $\mathbf{m}(A) = \mathbf{m}$, then A does not have the SSP, yet there exists $B \in \mathcal{S}(G)$ with $\mathbf{m}(B) = \mathbf{m}$ that has the SMP. To establish that for a given graph and ordered multiplicity list, no matrix can have the SSP, we apply the next result.

Theorem 4.1. [8, Corollary 29(b)] *Suppose G is a graph, $A \in \mathcal{S}(G)$, $\mathbf{m}(A) = (m_1, \ldots, m_q)$, and A has the SSP. Then $|E(G)| \geq \sum_{j=1}^{q} \binom{m_i}{2}$.*

Proposition 4.2. *Let H be the graph shown in Figure 2. If $A \in \mathcal{S}(H)$ and $\mathbf{m}(A) = (3, 5, 4)$, then A does not have the SSP. The matrix*

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & -1 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & -1 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 & 0 & 2 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & \sqrt{2} & 0 & 0 & -2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 \end{pmatrix} \in \mathcal{S}(H)$$

has $\operatorname{spec}(B) = \{-4, -4, -4, 0, 0, 0, 0, 0, 3, 3, 3, 3\}, \mathbf{m}(B) = (3, 5, 4)$*, and B has the SMP.*

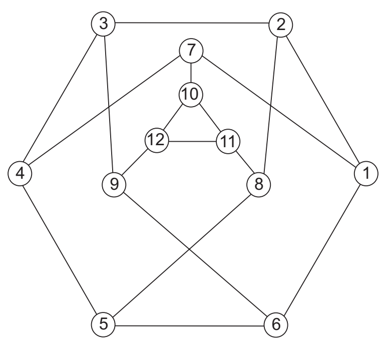

Figure 2: The graph H, which allows ordered multiplicity list (3,5,4) with the SMP but not with the SSP.

Proof. Consider a matrix $A \in \mathcal{S}(H)$ with $\mathbf{m}(A) = (3, 5, 4)$. Since $\binom{3}{2} + \binom{5}{2} + \binom{4}{2} = 19 > 18 = |E(H)|$, A does not have the SSP by Theorem 4.1. It is straightforward to verify computationally that B does have the SMP (see [23]) and has the stated spectrum. $\square$

5 Minimal minors for multiple eigenvalues

An eigenvalue is *multiple* if it is not simple, i.e., if it has multiplicity at least two. In this section we determine the forbidden minors for a graph to have at most one multiple eigenvalue in a matrix with the SSP or the SMP, and characterize connected graphs that do not have two consecutive multiple eigenvalues.

5.1 Minimal minors having at least two multiple eigenvalues

Theorem 5.1. *Let G be a graph.*

(1) If G is a connected graph and none of the eleven graphs shown in Figure 3 is a minor of G, then any matrix $A \in \mathcal{S}(G)$ has at most one multiple eigenvalue.

(2) There exists a matrix $A \in \mathcal{S}(G)$ with SSP (or with the SMP) having more than one multiple eigenvalue if and only if one of the eleven graphs shown in Figure 3 is a minor of G.

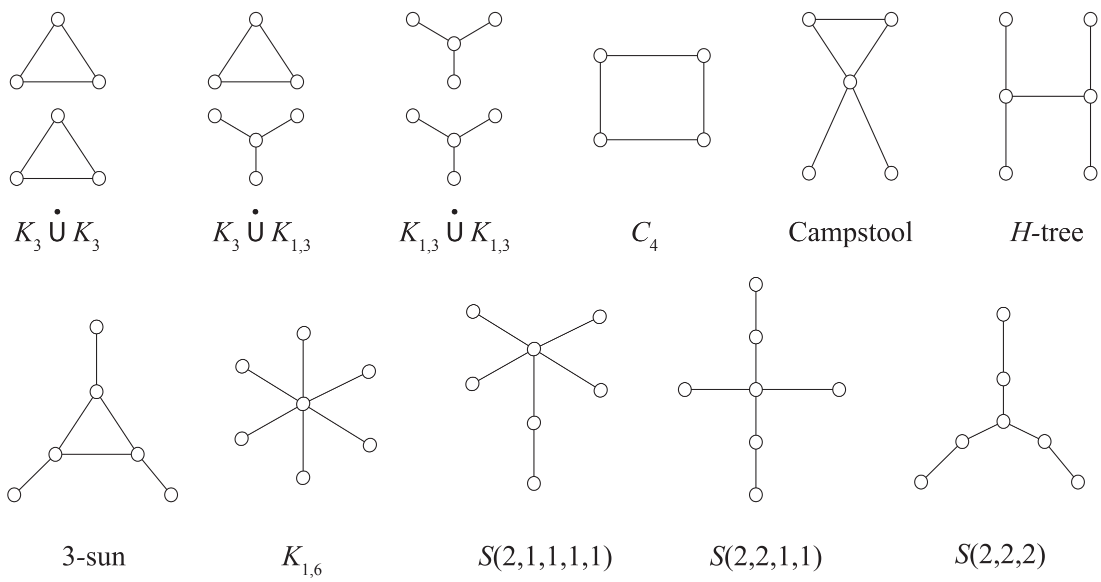

Figure 3: The eleven minimal minors for two multiple eigenvalues.

Proof. (1) Suppose that G is a connected graph that has none of these eleven graphs as a minor. We show that $A \in \mathcal{S}(G)$ can have at most one multiple eigenvalue. The statement is immediate if $|G| \leq 3$ so assume $|G| \geq 4$.

Case A: G contains a cycle. Since C_4 is not a minor of G, the only cycles in G are triangles. Moreover, there can be no pair of disjoint triangles in G since $K_3 \dot\cup K_3$ is not a minor of G, there can be no pair of triangles intersecting in exactly a single vertex since Campstool is not a minor of G, and there can be no pair of triangles intersecting in exactly a single edge since C_4 is not a minor of G. Thus there is a single 3-cycle C in G. Since the 3-sun is not a minor of G, one of the vertices of C has degree 2, and since Campstool is not a minor of G, the degrees of the other two vertices of C are at most 3. Since $|G| \geq 4$, at least one of the vertices of C has degree 3. Then all vertices not on C must have degree at most 2 or else the H-tree would be a minor of G. It follows that G is a path with an additional edge joining two vertices on the path at distance 2. By [8, Proposition 50 and Theorem 51], $q(G) = n - 1$ and A has at most a single multiple eigenvalue.

Case B: G is a tree. Since the H-tree is not a minor of G, G has at most one vertex v of degree greater than 2. If A has two distinct eigenvalues of multiplicity at least 2, then by the Parter–Wiener Theorem (Theorem 2.1), $A[G-v]$ has two eigenvalues of multiplicity at least 3 and each of these must occur in at least 3 different components of $G-v$. We show this is impossible.

The degree of v is at most 5 since $K_{1,6}$ is not a minor of G. If the degree of v is 5, then G is $K_{1,5}$ since $S(2,1,1,1,1)$ is not a minor of G, but this does not permit 3 components each for 2 eigenvalues. Suppose the degree of v is 4. Since $S(2,2,1,1)$ is not a minor of G, all but one of the neighbors of v is a pendent vertex, which also does not permit 3 components each for 2 eigenvalues. Suppose the degree of v is 3. Since $S(2,2,2)$ is not a minor, at least one neighbor of v is pendent, but this does not permit 3 components each with 2 eigenvalues. The only remaining case is that G is a path, and then all eigenvalues of A are simple.

(2) ($\Rightarrow$): Suppose that G does not have any of these eleven graphs as a minor. The case in which G is connected is covered by (1), so assume G is disconnected. Since $K_3 \dot\cup K_3$ is not a minor, G has at most one component with a cycle. Let $A \in \mathcal{S}(G)$ have the SMP.

Table 3: Matrices with two double eigenvalues and the SSP

graph	matrix/reason	spectrum	OML
$K_3 \dot\cup K_3$	Remark 3.1		(2,2,1,1)
$K_3 \dot\cup K_{1,3}$	Remark 3.1		$(1,2,2,1,1)$
$K_{1,3} \dot\cup K_{1,3}$	Remark 3.1		$(1,1,2,2,1,1)$
C_4	Table 1	$\{-\sqrt{2},-\sqrt{2},\sqrt{2},\sqrt{2}\}$	$(2,2)$
Campstool	[8, Proposition 44]	$\{-2,0,0,2,2\}$	$(1,2,2)$
H-tree	[8, Proposition 44]	$\{\frac{1}{2}(1-\sqrt{29}),0,0,1,1,\frac{1}{2}(1+\sqrt{29})\}$	$(1,2,2,1)$
3-sun	[8, Proposition 44]	$\{0,0,\frac{1}{2}(5-\sqrt{13}),2,2,\ \frac{1}{2}(5+\sqrt{13})\}$	$(2,1,2,1)$
$K_{1,6}$	$\begin{pmatrix} 0&1&1&1&2&2&2\\ 1&1&0&0&0&0&0\\ 1&0&1&0&0&0&0\\ 1&0&0&1&0&0&0\\ 2&0&0&0&0&0&0\\ 2&0&0&0&0&0&0\\ 2&0&0&0&0&0&0 \end{pmatrix}$	$\{\frac{1}{2}(-3-\sqrt{21}),\ 0,0,\ \frac{1}{2}(-3+\sqrt{21}),1,1,4\}$	$(1,2,1,2,1)$
$S(2,1,1,1,1)$	$\begin{pmatrix} 0&1&0&3&2&1&1\\ 1&1&1&0&0&0&0\\ 0&1&1&0&0&0&0\\ 3&0&0&2&0&0&0\\ 2&0&0&0&2&0&0\\ 1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0 \end{pmatrix}$	$\{\frac{1}{2}(-3-\sqrt{13}),\ 0,0,\ \frac{1}{2}(\sqrt{13}-3),2,2\}$	$(1,2,1,2,1)$
S(2, 2, 1, 1)	$\begin{pmatrix} 0&2&0&2&0&1&\sqrt{2}\\ 2&1&1&0&0&0&0\\ 0&1&1&0&0&0&0\\ 2&0&0&1&1&0&0\\ 0&0&0&1&1&0&0\\ 1&0&0&0&0&2&0\\ \sqrt{2}&0&0&0&0&0&0 \end{pmatrix}$	$\{-3,0,0,1,2,2,4\}$	$(1,2,1,2,1)$
S(2,2,2)	[8, Proposition 44]	$\{-1.4605,0,0,.760877,2,2,2.69963\}$	$(1,2,1,2,1)$

Case A: G has a component that contains a cycle. Call this component G_1. Since $K_3 \dot\cup K_{1,3}$ is not a minor of G, all other components of G are paths. By [8, Theorem 34], the spectra associated with the components must be disjoint, so all multiple eigenvalues must be associated with the connected component G_1. Then by (1), all but one of the eigenvalues associated with G_1 are simple.

Case B: G is a forest. Since $K_{1,3} \dot\cup K_{1,3}$ is not a minor, all but possibly one component are paths; denote the non-path component by T_1. Since the spectra associated with the components must be disjoint, all multiple eigenvalues must be associated with the component T_1, and by (1), all but one of the eigenvalues associated with T_1 are simple.

(2) ($\Leftarrow$): For each graph in Figure 3, Table 3 lists one of the following: i) A matrix in $\mathcal{S}(G)$ (or a citation of a reference that contains such a matrix) together with its eigenvalues and ordered multiplicity list; the matrix has two eigenvalues of multiplicity 2 and has the SSP. ii) A reason that implies the graph has a matrix with two multiple eigenvalues and the SSP. Then, by Theorem 6.13, any graph that has one of these eleven graphs as a minor must allow a matrix with the SSP that has at least two multiple eigenvalues. $\square$

5.2 Minimal minors having at least two consecutive multiple eigenvalues

Let $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ be the distinct eigenvalues of a matrix. Eigenvalues of the form $\lambda_i, \lambda_{i+1}, \ldots, \lambda_{i+s}$ with $s \geq 1$ are referred to as *consecutive*. Theorem 5.3 below characterizes, by forbidden minors, the graphs that do not allow two consecutive multiple eigenvalues. To prove this theorem, we need some additional results and notation. If $\mathcal{F}$ is a family of graphs, we say that a graph G *does not have an $\mathcal{F}$-minor* if no graph in $\mathcal{F}$ is a minor of G.

Examining Table 3, some of the eleven graphs in Figure 3 allow a matrix with the SSP and two consecutive multiple eigenvalues, including

$$\{K_3 \dot\cup K_3, K_3 \dot\cup K_{1,3}, K_{1,3} \dot\cup K_{1,3}, C_4, \text{Campstool}, H\text{-tree}\}.$$

The remaining graphs are 3-sun, $K_{1,6}, S(2,1,1,1,1), S(2,2,1,1), S(2,2,2)$; all but the 3-sun are generalized stars.

A *generalized* 3*-sun* is obtained from the 3-sun by subdividing each edge incident with a vertex of degree one as many times as desired (note no subdivision is acceptable, so the 3-sun is also a generalized 3-sun). It is well known that a generalized star G does not allow a matrix with two consecutive multiple eigenvalues (Lemma 2.3), and we show this is also a property of a generalized 3-sun.

Lemma 5.2. *A generalized* 3*-sun does not allow a matrix with two consecutive multiple eigenvalues.*

Proof. Suppose $A \in \mathcal{S}(G)$ has consecutive multiple eigenvalues. Since G is unicyclic with an odd cycle, we may assume all off diagonal entries have the same sign. By shifting and scaling we may also assume the two consecutive multiple eigenvalues are 1 and -1, so $A^2 - I$ is a positive semidefinite matrix with nullity at least 4.

Define the graphs $G^{(2)}$ and G^2 by $V(G^{(2)}) = V(G^2) = V(G)$, $E(G^{(2)})$ contains all pairs of vertices that have distance 2 in G, and $E(G^2) = E(G) \dot\cup E(G^{(2)})$. Let H be the graph corresponding to the matrix $A^2 - I$. Then $E(G^{(2)}) \subseteq E(H) \subseteq E(G^2)$. Now let S be the three vertices on the center triangle of G. Then S is a positive semidefinite zero forcing set of H, meaning $M_+(H) \leq Z_+(H) \leq 3$, which is a contradiction. For the definition of $Z_+(G)$ and its relation with $M_+(G)$, see [3] or [14]. $\square$

Theorem 5.3. *The following statements are equivalent:*

1. *G does not allow a matrix with the SSP that has two consecutive multiple eigenvalues;*
2. *G does not allow a matrix with the SMP that has two consecutive multiple eigenvalues;*
3. *G does not contain a minor in the family*

$$\mathcal{F}_2' = \{K_3 \dot\cup K_3, K_3 \dot\cup K_{1,3}, K_{1,3} \dot\cup K_{1,3}, C_4, \text{Campstool}, H\text{-tree}\};$$

4. *G is a disjoint union of G_1 and any number of paths, where G_1 is either a generalized star or a generalized 3-sun.*

Proof. For graphs G in $\mathcal{F}_2'$, there is a matrix $A \in \mathcal{S}(G)$ with the SSP and having consecutive multiple eigenvalues (see Table 3). Then by Theorem 6.13, every graph that contains a $\mathcal{F}_2'$ minor allows a matrix with the SSP and having consecutive multiple eigenvalues. Such a matrix also has the SMP. In other words, (2) $\Rightarrow$ (1) $\Rightarrow$ (3).

To see that (3) $\Rightarrow$ (4), assume G is a graph that does not contain a $\mathcal{F}_2'$ minor. Since G does not have any of $K_3 \dot\cup K_3$, $K_3 \dot\cup K_{1,3}$, or $K_{1,3} \dot\cup K_{1,3}$ as a minor, every component of G but one is a path. Since paths do not allow multiple eigenvalues and the SSP guarantees eigenvalues from different components are not the same, we may assume G is connected. Since G does not have any of $C_4, K_3 \dot\cup K_3$, or Campstool as a minor, G is either a tree or a unicyclic graph with a triangle.

Suppose G is unicyclic with a triangle. Consider G formed by attaching three trees T_1, T_2, T_3 to a triangle on vertices v_1, v_2, v_3 respectively. If T_i is not a path with v_i as an endpoint, then G has the Campstool as a minor. Thus G is a generalized 3-sun, and G does not allow a matrix having consecutive multiple eigenvalues by Lemma 5.2.

Suppose G is a tree. If G has two vertices of degree at least 3, then G has the H-tree as a minor. So G must be a generalized star, which does not allow a matrix having consecutive multiple eigenvalues by Lemma 2.3.

The fact that (4) implies (2) is obvious. □

Corollary 5.4. *A graph G allows a matrix with two consecutive multiple eigenvalues and the SSP if and only if G allows a matrix with two consecutive multiple eigenvalues and the SMP if and only if G has a minor in the family $\mathcal{F}_2' = \{K_3 \dot\cup K_3, K_3 \dot\cup K_{1,3}, K_{1,3} \dot\cup K_{1,3}, C_4, \text{Campstool}, H\text{-tree}\}$.*

If a graph G does not allow a matrix with two consecutive multiple eigenvalues, then G does not allow a matrix with the SSP and having two consecutive multiple eigenvalues. Thus, we have the following corollary to Theorem 5.3 and Lemma 5.2.

Corollary 5.5. *A connected graph G does not allow a matrix with two consecutive multiple eigenvalues if and only if G is a generalized star or a generalized 3-sun.*

6 Minor monotonicity of the SSP and SMP

It is known that the maximum nullity of a graph is not minor monotone (in fact, not even subgraph monotone [6, Example 5.1]). However, under the additional assumption of having the SAP the maximum nullity is minor monotone; that is, if G is a minor of the graph H and there is a matrix in $\mathcal{S}(G)$ with nullity k and the SAP, then there is a matrix in $\mathcal{S}(H)$ with nullity k and the SAP [7].

In this section we study minor monotonicity with respect to spectra and multiplicity lists and the SSP and SMP properties. Note that the monotonicity of SSP and SMP for G a subgraph of H was established in [8]. However, contraction is more subtle. When G is obtained from H by a contraction, it may be necessary to restrict the simple eigenvalue added to being at one of the extreme ends of the spectrum. For example, C_4 is a minor of C_5, C_4 realizes (2,2) with the SSP, but C_5 cannot realize (2,1,2) by Lemma 3.3, yet C_5 can realize (2,2,1). Much of our discussion will focus on the SSP since the proofs for the SMP are analogous.

6.1 Tangent spaces of pertinent manifolds

One of the key ideas used to prove the minor monotonicity is to perturb a matrix so that it stays on the manifold of matrices with the same spectrum, ordered multiplicity list, or rank. Consider the space of all $n \times n$ symmetric real matrices, denoted as $S_n(\mathbb{R})$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a set of real numbers, $\mathbf{m} = (m_1, \ldots, m_q)$ a sequence of positive integers with $m_1 + \cdots + m_q = n$, r an integer with $0 \le r \le n$. Define

$$\begin{aligned} \mathcal{E}_\Lambda &= \{M \in S_n(\mathbb{R}) : \operatorname{spec}(M) = \Lambda\}, \\ \mathcal{U}_{\mathbf{m}} &= \{M \in S_n(\mathbb{R}) : \mathbf{m}(M) = \mathbf{m}\}, \text{ and} \\ \mathcal{R}_r &= \{M \in S_n(\mathbb{R}) : \operatorname{rank}(M) = r\}. \end{aligned}$$

For any matrix $A \in S_n(\mathbb{R})$, we also write $\mathcal{E}_A$ for $\mathcal{E}_{\operatorname{spec}(A)}$, $\mathcal{U}_A$ for $\mathcal{U}_{\mathbf{m}(A)}$, and $\mathcal{R}_A$ for $\mathcal{R}_{\operatorname{rank}(A)}$. The notation follows that in [8], and these sets are known to be manifolds; see, e.g., [10]. #5

Recall that two manifolds *intersect transversally* at a point $\mathbf{x}$ if their tangent spaces at $\mathbf{x}$ span the whole space; equivalently, two manifolds intersect transversally if the intersection of their normal spaces is $\{\mathbf{0}\}$. Also, if G is a graph and $A \in \mathcal{S}(G)$, then Lemmas 4, 7, and 17 in [8] assert that the $\mathcal{S}(G)$ and $\mathcal{E}_A$ ($\mathcal{U}_A$, or $\mathcal{R}_A$, respectively) intersect transversally at A if and only if A has the SSP LH (SMP, or SAP, respectively). The tangent spaces for $\mathcal{E}_A$, $\mathcal{U}_A$, and $\mathcal{R}_A$ at the point A have been computed in [8, Theorem 27]. To utilize these tangent spaces in an efficient way, Definition 6.1 introduces the tangent space matrices so that these tangent spaces are the column spaces of the corresponding matrices. #4 (1006)

For an $n \times n$ symmetric matrix M, let $\operatorname{vec}(M)$ be the vector of dimension $\binom{n+1}{2}$ with entries from the upper triangular part of M; $\operatorname{vec}(M)$ is indexed by (i,j) in lexicographic order for $1 \le i \le j \le n$. For a set E of pairs (i,j) with $1 \le i \le j \le n$, $\operatorname{vec}_E(M)$ is the subvector of dimension $|E|$ that contains only the entries corresponding to indices in E. In the following, E_{ij} denotes the $n \times n$ matrix with a 1 in position (i,j) and 0 elsewhere, and K_{ij} denotes the $n \times n$ skew-symmetric matrix $E_{ij} - E_{ji}$.

Definition 6.1. Let M be an $n \times n$ symmetric matrix.

1. The *SSP tangent space matrix* $\mathrm{TS}_{\mathrm{S}}(M)$ of M is the $\binom{n+1}{2} \times \binom{n}{2}$ matrix such that the (k,ℓ)-column is $\operatorname{vec}(MK_{k\ell} + K_{\ell k}M)$ and the columns are indexed by (k,ℓ) in lexicographic order for $1 \le k < \ell \le n$.

2. The *SMP tangent space matrix* $\mathrm{TS}_{\mathrm{M}}(M)$ of M is the $\binom{n+1}{2} \times \left(\binom{n}{2} + q\right)$ matrix obtained by augmenting $\mathrm{TS}_{\mathrm{S}}(M)$ with the q columns $\operatorname{vec}(M^k)$ for $k = 0, 1, \ldots, q-1$ (where q is the number of distinct eigenvalues of M).

3. The *SAP tangent space matrix* $\mathrm{TS}_{\mathrm{A}}(M)$ of M is the $\binom{n+1}{2} \times n^2$ matrix such that the (k,ℓ)-column is $\operatorname{vec}(ME_{k\ell} + E_{\ell k}M)$ and the columns are indexed by (k,ℓ) with $1 \le k, \ell \le n$ in the lexicographic order.

something commented out here.

Remark 6.2. Let A be a symmetric matrix and $\mathcal{E}_A$ its isospectral manifold. Let V be a subspace of symmetric matrices. **According to the definition of the transversality**, $\mathcal{E}_A$ and V intersect transversally at A if and only if the zero matrix is the only matrix X such that $X \in V^\perp$ and $\operatorname{vec}(X)^\top \operatorname{TS_S}(A) = \mathbf{0}^\top$. In particular, if G is the graph of A and V is the **topological closure of** $\mathcal{S}(G)$, then we can see that A has the SSP if and only if the rows of $\operatorname{TS_S}(A)$ corresponding to nonedges are linearly independent. #6

We now focus on the tangent space to the isospectral manifold. We first compute the tangent space matrix for a matrix of the form $A \oplus [\lambda]$, which is denoted by A_λ. Given a matrix M, $\max(M)$ denotes the maximum absolute value of an entry of M, and $\mathbf{m}_j$ denotes the jth column of M.

Lemma 6.3. *Let A be an $n \times n$ matrix and λ be a real number. Then, after the columns indexed by $(i, n+1)$ have been permuted to the right and the rows indexed by $(i, n+1)$ have been permuted to the bottom,* $\operatorname{TS_S}(A_\lambda)$ *has the form*

$$\left(\begin{array}{ccc|ccc} & \operatorname{TS_S}(A) & & & O & \\ \hline & O & & & A - \lambda I_n & \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right).$$

Proof. The result follows from the following computations. For $i, j \in \{1, 2, \ldots, n\}$,

$$A_\lambda K_{ij} - K_{ij} A_\lambda = \left(\begin{array}{ccc|c} & & & 0 \\ & AK_{ij} - K_{ij}A & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right).$$

For $i \in \{1, 2, \ldots, n\}$,

$$A_\lambda K_{i,n+1} - K_{i,n+1} A_\lambda = \left(\begin{array}{c|c} O & \mathbf{a}_i - \lambda \mathbf{e}_i \\ \hline \mathbf{a}_i^\top - \lambda \mathbf{e}_i^\top & 0 \end{array}\right),$$

where $\mathbf{a}_i$ and $\mathbf{e}_i$ denote the ith column of A and the ith standard basis vector, respectively. □

We now use Lemma 6.3 to give a perturbation result for A_λ. Specifically, if λ is large enough, you can perturb an SSP matrix A_λ by any sufficiently small E that is combinatorially orthogonal to A_λ, and find a correction matrix F whose size is controlled such that $\mathcal{G}(F)$ is a subgraph of $\mathcal{G}(A_\lambda)$ and $\operatorname{spec}(A_\lambda + E + F) = \operatorname{spec}(A_\lambda)$. This allows us to add edges between the isolated vertex represented by λ and $\mathcal{G}(A)$ while preserving the spectrum and controlling the size of the modification.

Before continuing the argument, we include a version [8, Theorem 3] of the implicit function theorem regarding the intersection of two manifolds.

Theorem 6.4. [8] *Let $\mathcal{M}_1(s)$ and $\mathcal{M}_2(t)$ be smooth families of manifolds in $\mathbb{R}^d$. Suppose $\mathcal{M}_1(0)$ and $\mathcal{M}_2(0)$ intersect transversally at a point $\mathbf{y}_0$. Then there is a neighrborhood $W \subseteq \mathbb{R}^2$ of the origin and a continuous function $f : W \to \mathbb{R}^d$ such that $f(\mathbf{0}) = \mathbf{y}_0$ and for each $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2) \in W$, $\mathcal{M}_1(\epsilon_1)$ and $\mathcal{M}_2(\epsilon_2)$ intersect transversally at $f(\boldsymbol{\epsilon})$.*

#7

Lemma 6.5. *Let A be an $n \times n$ symmetric matrix with the SSP and graph G. There exists a $\Delta > 0$ such that for all sufficiently small $\epsilon > 0$ and all sufficiently large λ, the following holds:*

For each symmetric matrix E of order $n+1$ such that $E \circ I = O$ and $E \circ A_\lambda = O$ and $\max(E) \le \epsilon$, there exists a symmetric $F = [f_{ij}]$ of order $n+1$ such that $f_{ij} \neq 0$ only if $i = j$ or ij is an edge of G, $\max(F) \le \Delta\epsilon$, and $A_\lambda + E + F$ is cospectral with A_λ. #3

Proof. For λ sufficiently large, λ is not an eigenvalue of A, so A_λ has the SSP, and $\mathcal{G}(A_\lambda) = G \,\dot\cup\, K_1$. Let τ be the indices of the rows of $\mathrm{TS}_\mathrm{S}(A_\lambda)$ corresponding to (i,j) where $i \neq j$ and ij is not an edge of $\mathcal{G}(A_\lambda)$. Since A_λ has the SSP, the rows of $\mathrm{TS}_\mathrm{S}(A_\lambda)$ corresponding to τ are linearly independent and there exists an invertible $|\tau| \times |\tau|$ submatrix $\mathrm{TS}_\mathrm{S}(A_\lambda)[\tau,\mu]$ of $\mathrm{TS}_\mathrm{S}(A_\lambda)$.

We are building toward a proof that the SSP is preserved for decontractions. The proof that the SSP is preserved for supergraphs [8, Theorem 10] makes use of the Implicit Function Theorem. For this result, we need a **different form** of the Implicit Function Theorem, which uses the invertibility of $\mathrm{TS}_\mathrm{S}(A_\lambda)[\tau,\mu]$ to guarantee uniform continuity within some neighborhood. This yields, for $\epsilon > 0$ sufficiently small and a given E with $\max(E) \le \epsilon$, a positive Δ (independent of ϵ) and F satisfying the conditions stated in the lemma, for which $A_\lambda + E + F$ cospectral with A_λ. □

need explanation (1006)

Next we compute the tangent space of a special type of perturbation of A_λ. Recall that the notation $O(\epsilon)$ denotes a function $f(\epsilon)$ with the property that there is a constant $K > 0$ and a small number $\delta > 0$ such that $|f(\epsilon)| < K\epsilon$ and $|f(-\epsilon)| < K\epsilon$ whenever $\epsilon < \delta$. #8

Lemma 6.6. *Given A_λ and a matrix L with $\max(L) \le 1$, suppose*

$$C = A_\lambda + \left(\begin{array}{c|c} O & \mathbf{x} \\ \hline \mathbf{x}^\top & 0 \end{array}\right) + O(\epsilon)L.$$

Then, after permuting as in Lemma 6.3, $\mathrm{TS}_\mathrm{S}(C)$ has the form

$$\left(\begin{array}{c|c} \mathrm{TS}_\mathrm{S}(A) & V_1 \\ \hline V_2 & A - \lambda I + V_3 \\ \hline \mathbf{0}^\top & 2\mathbf{x}^\top \end{array}\right) + O(\epsilon)\,\mathrm{TS}_\mathrm{S}(L) = \left(\begin{array}{c|c} \mathrm{TS}_\mathrm{S}(A) & V_1 \\ \hline V_2 & A - \lambda I + V_3 \\ \hline \mathbf{0}^\top & 2\mathbf{x}^\top \end{array}\right) + O(\epsilon)M$$

where $\max(V_1), \max(V_2), \max(V_3)$ are all at most $2\max(\mathbf{x})$, and M is a fixed matrix.

Proof. The form of C implies

$$\mathrm{TS}_\mathrm{S}(C) = \mathrm{TS}_\mathrm{S}(A_\lambda) + \mathrm{TS}_\mathrm{S}\left(\left(\begin{array}{c|c} O & \mathbf{x} \\ \hline \mathbf{x}^\top & 0 \end{array}\right)\right) + O(\epsilon)\,\mathrm{TS}_\mathrm{S}(L).$$

The result then follows from the following computations. For $i, j \in \{1, 2, \ldots, n\}$,

$$\left(\begin{array}{c|c} O & \mathbf{x} \\ \hline \mathbf{x}^\top & 0 \end{array}\right) K_{ij} + K_{ji} \left(\begin{array}{c|c} O & \mathbf{x} \\ \hline \mathbf{x}^\top & 0 \end{array}\right) = \left(\begin{array}{c|c} O & x_i\mathbf{e}_j - x_j\mathbf{e}_i \\ \hline x_i{\mathbf{e}_j}^\top - x_j{\mathbf{e}_i}^\top & 0 \end{array}\right).$$

For $i \in \{1, 2, \ldots, n\}$,

$$\left(\begin{array}{c|c} O & \mathbf{x} \\ \hline \mathbf{x}^\top & 0 \end{array}\right) K_{i,n+1} + K_{n+1,i} \left(\begin{array}{c|c} O & \mathbf{x} \\ \hline \mathbf{x}^\top & 0 \end{array}\right) = \left(\begin{array}{c|c} -\mathbf{e}_i\mathbf{x}^\top - \mathbf{x}{\mathbf{e}_i}^\top & O \\ \hline O & 2x_i \end{array}\right).$$ □

Although it is not part of the statement of the next result, we are thinking of a graph G of order n being used to create a graph H of order $n+1$ as follows: Add a new vertex $n+1$ **adjacent to** n. All the edges of G that do not involve vertex n remain in H. Partition the neighborhood of n in G, $N_G(n)$, into α and β. In H, the vertices in α remain adjacent to n whereas the vertices in β are adjacent to $n+1$ but not to n. Thus H has one more vertex and one more edge than G. The existence of a matrix of the special form described in Theorem 6.7 ensures the existence of a matrix in $\mathcal{S}(H)$ with spectrum $\operatorname{spec}(G) \dot\cup \{\lambda\}$ for λ sufficiently large. #9-(1006-LH)

Theorem 6.7. *Let A be an $n \times n$ SSP matrix with graph G, λ sufficiently large, and $\alpha \dot\cup \beta$ be a partition of $N_G(n)$. Suppose that for a matrix L with $\max(L) \le 1$, there is a symmetric $(n+1) \times (n+1)$ matrix*

$$C = A_\lambda + \left(\begin{array}{c|c} O & \mathbf{x} \\ \hline \mathbf{x}^\top & 0 \end{array} \right) + O(\epsilon)L,$$

such that $\operatorname{spec}(C) = \operatorname{spec}(A_\lambda)$,

$$\begin{array}{ll} c_{n,j} = c_{n+1,j} = 0 & \text{for } j \notin N_G[n] \cup \{n+1\}, \\ c_{n,j} = c_{n+1,j} & \text{for } j \in \alpha \cup \{n\}, \\ c_{n,j} = -c_{n+1,j} & \text{for } j \in \beta, \\ c_{n,n} \neq c_{n+1,n+1}, & \end{array}$$

and $C(\{n+1\})$ has graph G, where $C(\{n+1\})$ is the matrix obtained from C by removing the last column and row. Let V denote the span of the matrices #10 (1006)

$$\begin{array}{cl} E_{ii} & \text{for } i = 1, 2, \ldots, n+1, \\ E_{ij} + E_{ji} & \text{for } ij \text{ an edge of } G \text{ not incident to } n, \\ E_{nj} + E_{jn} - E_{n+1,j} - E_{j,n+1} & \text{for } j \in \alpha \cup \{n\}, \text{ and} \\ E_{nj} + E_{jn} + E_{n+1,j} + E_{j,n+1} & \text{for } j \in \beta. \end{array}$$

*Then $\mathcal{E}_C$ intersects V **transversally**. Moreover, the matrix $RCR^\top$, where* typo

$$R = I_{n-1} \oplus \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix},$$

is cospectral with A_λ, has the SSP, and its graph H is the graph obtained from G by inserting a new vertex $n+1$, and edge between n and $n+1$, and for $j \in \beta$ replacing the edge nj in G with the edge joining j and $n+1$.

Proof. According to Remark 6.2, the two manifolds $\mathcal{E}_C$ and V intersect transversally if and only if $X = O$ is the only matrix with $X \in V^\perp$ and $\operatorname{vec}(X)^\top \operatorname{TS_S}(C) = \mathbf{0}^\top$. Let $\mathbf{w} = \operatorname{vec}(X)$ for some matrix $X \in V^\perp$ and assume that $\mathbf{w}$ is in the left-nullspace of $\operatorname{TS_S}(C)$. For convenience we index the elements of $\mathbf{w}$ elements by (i,j) in lexicographic order with $1 \le i \le j \le n+1$. Thus we need to show that if $\mathbf{w}_{(i,i)} = 0$ for all i, $\mathbf{w}_{(i,j)} = 0$ when ij is an edge of G not incident to n, $\mathbf{w}_{(j,n)} = \mathbf{w}_{(j,n+1)}$ for $j \in \alpha \cup \{n\}$, and $\mathbf{w}_{(j,n)} = -\mathbf{w}_{(j,n+1)}$ for $j \in \beta$, then $\mathbf{w} = \mathbf{0}$.

Let $\widehat{C}$ be the matrix obtained from $\operatorname{TS_S}(C)$ by the following three steps:

- Replace the row indexed by $(j, n+1)$ by the sum of the rows indexed by $(j, n+1)$ and (j, n) for $j \in \alpha \cup \{n\}$.

- Replace the row indexed by $(j, n+1)$ by the difference of the rows indexed by $(j, n+1)$ and (j, n) for $j \in \beta$.

Thus, the desired transversality is implied by the linearly independence of rows of $\widehat{C}$ indexed by $\gamma = \{(i,j) : 1 \le i < j \le n+1 \text{ and } ij \text{ not an edge of } G\}$ and the rows indexed by $(j, n+1)$ for $j = 1, \ldots, n$. By Lemma 6.6, after the usual permutation, the matrix formed by these rows has the form #11 (1006)

$$\left(\begin{array}{c|c} \mathrm{TS}_{\mathrm{S}}(A)[\gamma, :] & V_1[\gamma, :] \\ \hline V_2' & A - \lambda I_n + V_3' \end{array}\right) + O(\epsilon)M \tag{1}$$

for a fixed matrix M. Since λ is significantly larger than any entry of V_1, V_2', and V_3', and the corresponding rows of $\mathrm{TS}_{\mathrm{S}}(A)$ are linearly independent by the fact that A has the SSP, the matrix in (1) has linearly independent rows.

Note that $RCR^\top$ is the matrix obtained from C by performing a rotation of $\pi/4$ on the last two rows and columns of C. Hence the submatrices of $RCR^\top$ and C lying in the first $n-1$ rows and columns agree. The hypothesis on the last two rows of C imply that $RCR^\top$ has the given graph. #12

Now suppose that X is an $(n+1) \times (n+1)$ matrix with $X \circ I = O$, $X \circ RCR^\top = O$, and $[X, RCR^\top] = O$. Note in particular, $X[\{n, n+1\}, \{n, n+1\}] = O$ since $(RCR^\top)_{n,n+1} \neq 0$. It follows that $RXR^\top$ is in the orthogonal complement of V in the space of symmetric matrices. As V intersects $\mathcal{E}_C$ **transversally**, we conclude that $RXR^\top = O$. Thus, $X = O$, and we conclude that $RCR^\top$ has the SSP. □ typo

6.2 Proof of minor monotonicity for SSP

In this section we prove that if A has the SSP and λ is sufficiently large, then there exists a matrix C satisfying the hypothesis of Theorem 6.7. We begin with a needed perturbation result.

Lemma 6.8. *Let M be a positive definite $n \times n$ matrix with smallest eigenvalue $\lambda_{\min}(M)$ and $\mathbf{b}$ be an n-vector. If $\mathbf{x}$ is the solution to $M\mathbf{x} = \mathbf{b}$, then*

$$\max(\mathbf{x}) \le \frac{\sqrt{n}}{\lambda_{\min}(M)} \cdot \max(\mathbf{b}).$$

Proof. Let $\mathbf{m}_j$ denote the jth column of M^{-1}, and observe that $\|\mathbf{m}_j\| \le \lambda_{\max}(M^{-1}) = \frac{1}{\lambda_{\min}(M)}$. Since $\mathbf{x} = M^{-1}\mathbf{b}$, the Cauchy–Schwarz inequality yields

$$\begin{aligned} |x_j| &\le \|\mathbf{m}_j\| \cdot \|\mathbf{b}\| \\ &\le \frac{1}{\lambda_{\min}(M)} \cdot \|\mathbf{b}\| \\ &\le \frac{1}{\lambda_{\min}(M)} \cdot \sqrt{n} \cdot \max(\mathbf{b}). \end{aligned}$$

□

Proposition 6.9. *Let A be a symmetric $n \times n$ matrix with graph G. Let $\epsilon > 0$ with $\epsilon \le \max(A)$, $\Delta > 0$, and $\lambda \ge \max\left\{2\rho(A), \frac{4n(\max(A)(1+\Delta))^2}{\epsilon^2}\right\}$. Let $B = A_\lambda + E + F$ be an $(n+1) \times (n+1)$ matrix such that*

(a) $E = [e_{ij}]$ is a symmetric matrix with $\max(E) \le \epsilon$, $E \circ A_\lambda = O$, and $E \circ I = O$; and

(b) $F = [f_{ij}]$ is a symmetric matrix with $\max(F) \le \Delta \max(E)$ and $f_{ij} \neq 0$ only if ij is an edge of G or $i = j$.

Let $\alpha \dot\cup \beta$ be a partition of $N_G(n)$. Then there is an orthogonal matrix Q such that

(c) *for all i and j except $\{i,j\} \in \{\{n+1,k\} : k \in N_G[n]\}$, the ij-entry of $Q^\top BQ - B$ has absolute value at most $O(\epsilon^2)$.*

(d) *For each j in $\alpha \cup \{n\}$, $(Q^\top BQ)_{n+1,j} - (Q^\top BQ)_{nj} - e_{n+1,j}$ has absolute value at most $O(\epsilon^2)$, and*

(e) *for each j in β, $(Q^\top BQ)_{n+1,j} + (Q^\top BQ)_{nj} - e_{n+1,j}$ has absolute value at most $O(\epsilon^2)$.*

Proof. Since $\lambda \geq 2\rho(A)$, $\lambda I - A$ is positive definite. Define $\mathbf{k}$ to be the n-vector given by $(\lambda I - A)\mathbf{k} = D(\mathbf{a}_n + \mathbf{f}_n)$ where $\mathbf{a}_n$ and $\mathbf{f}_n$ denote the nth columns of A and $F(\{n+1\})$, and $D = \operatorname{diag}(d_1, \ldots, d_n)$ is the diagonal matrix with d_i equal to 1 if $i \in \alpha \cup \{n\}$, -1 if $i \in \beta$ and 0 otherwise.

Note that the choice of λ and Lemma 6.8 imply that

$$\begin{aligned}
\lambda \max(\mathbf{k})^2 &\leq \lambda \left(\frac{\sqrt{n}\max(A)(1+\Delta)}{\lambda_{\min}(\lambda I - A)} \right)^2 \\
&\leq \frac{\lambda n (\max(A)(1+\Delta))^2}{(\lambda - \rho(A))^2} \\
&\leq \frac{4\lambda n (\max(A)(1+\Delta))^2}{\lambda^2} \\
&\leq \epsilon^2.
\end{aligned}$$

Let K be the skew-symmetric matrix

$$\left(\begin{array}{c|c} O & -\mathbf{k} \\ \hline \mathbf{k}^\top & 0 \end{array} \right)$$

and let Q be the matrix exponential e^K, which is an orthogonal matrix. By the definition of matrix exponential and Taylor's theorem, e^K has the form $I + K + O(\epsilon^2)L'$ with $\max(L') \leq 1$. For ϵ #13 (1006)
sufficiently small, $Q^\top BQ$ has the form

$$B + A_\lambda K - KA_\lambda + O(\epsilon^2)L$$

with $\max(L) \leq 1$, because every entry of each of K, E, and F is $O(\epsilon)$. Since

$$A_\lambda K - KA_\lambda = \left(\begin{array}{c|c} O & D(\mathbf{a}_n + \mathbf{f}_n) \\ \hline (\mathbf{a}_n + \mathbf{f}_n)^\top D & O \end{array} \right),$$

$Q^\top BQ$ has the desired form. $\square$

Theorem 6.10. *Let A be a symmetric $n \times n$ matrix with graph G and the SSP, and let $\alpha \dot\cup \beta$ be a partition of $N_G(n)$. Let H be the graph obtained from $G \dot\cup K_1$ by joining $n+1$ to each vertex in $N_G[n]$. Then for $\epsilon > 0$ sufficiently small and λ sufficiently large there is a matrix C such that:*

- *the spectrum of C is that of A along with λ;*
- *$|c_{ij} - a_{ij}| \leq O(\epsilon)$ for all i and j with $\{i,j\} \notin \{\{n,j\} : j \in N_G[n]\}$;*
- *$c_{ni} = c_{n+1,i}$ for $i \in \alpha \cup \{n\}$;*

- $c_{ni} = -c_{n+1,i}$ *for* $i \in \beta$*;*
- $c_{n,n} \neq c_{n+1,n+1}$*; and*
- *the graph of* C *is* H*.*

Proof. Let Ω be the set of $(n+1)\times(n+1)$ symmetric matrices defined as

$$\Omega = \{E : \max(E) \le \epsilon, E \circ A_\lambda = O, \text{ and } E \circ I = O\}.$$

Since A has the SSP, by Lemma 6.5 there is a $\Delta > 0$ such that for $\epsilon > 0$ sufficiently small (we also require $\epsilon \le \max(A)$) and λ sufficiently large, for all $E \in \Omega$ there is a symmetric matrix F such that the graph of F is a subgraph of $G \dot{\cup} \{n+1\}$), $\max(F) \le \Delta \max(E)$, and $A_\lambda + E + F$ is cospectral to A_λ. As in Lemma 6.5, F can be chosen to be a uniformly continuous function of the entries of E. Denote such $A_\lambda + E + F$ by B_E.

With λ sufficiently large, $(B_E)_{nn} \neq (B_E)_{n+1,n+1}$. By Proposition 6.9, there exists a Q such that $Q^\top B_E Q$ satisfies (c), (d), and (e). Let $\phi(E)$ be the $(n+1)\times(n+1)$ symmetric matrix with its lower triangular part defined as

$$\phi(E)_{ij} = \begin{cases} (Q^\top B_E Q)_{ij} & \text{if } i \neq j,\ ij \text{ is not an edge of } H \\ (Q^\top B_E Q)_{n+1,j} - (Q^\top B_E Q)_{nj} & \text{if } i = n+1 \text{ and } j \in \alpha \cup \{n\} \\ (Q^\top B_E Q)_{n+1,j} + (Q^\top B_E Q)_{nj} & \text{if } i = n+1 \text{ and } j \in \beta \\ 0 & \text{otherwise.} \end{cases}$$

Then (c), (d), and (e) of Proposition 6.9, imply that

$$|\phi(E)_{ij} - e_{ij}| \le O(\epsilon^2) \text{ for all } ij \text{ with } i \neq j \text{ and } ij \text{ not an edge of } G \dot{\cup} \{n+1\},$$

where $E = [e_{ij}]$. We claim that there exists an E such that $\phi(E) = O$. Suppose to the contrary that $\phi(E) \neq O$ for all E. Then $f : \Omega \to \Omega$ **defined** by $f(E) = -\epsilon \frac{\phi(E)}{\max \phi(E)}$ is a well-defined, continuous #14
map. Let ij be an index with $|\phi(E)_{ij}|$ **the** largest. Note that e_{ij} and $f(E)_{ij}$ have opposite signs #15
unless $|e_{ij}| \le O(\epsilon^2)$, and in the latter case $|f(E)_{ij}| = \epsilon > |e_{ij}|$. Thus f has no fixed point. However, Ω is homeomorphic to a closed ball in $\mathbb{R}^d$, where d is the number of edges not in $G \dot{\cup} \{n+1\}$. So the nonexistence of a fixed point would contradict the Brouwer Fixed-Point Theorem.

Thus there exists $E \in \Omega$ such that $\phi(E) = O$. For such E, B_E, and hence $Q^\top B_E Q$, is cospectral with A_λ. Let $C = Q^\top B_E Q$. Then C has the desired properties.

□

Applying Theorem 6.7 to the matrix C found in Theorem 6.10, we have the following result.

Lemma 6.11 (Decontraction Lemma for SSP)**.** *Suppose* G *is obtained from* H *by contraction of a single edge whose endpoints have disjoint neighborhoods, and* $A \in \mathcal{S}(G)$ *with the SSP. Then for* λ *sufficiently large, there is an SSP matrix* $A' \in \mathcal{S}(H)$ *with the same eigenvalues as* A *and the additional eigenvalue* λ*.*

Similar arguments can be employed to establish the following analogous result for SMP.

Lemma 6.12 (Decontraction Lemma for SMP)**.** *Suppose* G *is obtained from* H *by contraction of a single edge whose endpoints have disjoint neighborhoods, and* $A \in \mathcal{S}(G)$ *with SMP. Then there is an SMP matrix* $A' \in \mathcal{S}(H)$ *with the ordered multiplicity list obtained by adding a 1 to the end of* $\mathbf{m}(A)$ *.*

Combining these results, and subgraph results [8, Theorem 36], we obtain the following general result regarding graph minors.

Theorem 6.13 (Minor Monotonicity Theorem)**.** *Suppose G is a minor of H obtained by contraction of r edges, deletion of s vertices, and deletion of any number of edges, and $A \in \mathcal{S}(G)$.*

If A has SMP and $\mathbf{m}(A) = (m_1, \ldots, m_t)$, then there is a matrix $A' \in \mathcal{S}(H)$ having SMP with $\mathbf{m}(A')$ obtained from $\mathbf{m}(A)$ by adding $r + s$ ones, with at most s of these between m_1 and m_t.

If in addition A has the SSP, then A' can be chosen to have the SSP, $\operatorname{spec}(A) \subseteq \operatorname{spec}(A')$*, all eigenvalues not in* $\operatorname{spec}(A)$ *are simple and distinct, at most s of these additional eigenvalues are between $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, and s simple eigenvalues (including all of those between $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$) can be chosen arbitrarily.*

7 The Matrix Liberation Lemma and other technical tools

In this section we prove the Matrix Liberation Lemma and some consequences, which were used to establish several previous results. In each case there is a rectangular matrix that characterizes the extent to which the zero entries of the matrix can be perturbed (with sufficiently small changes) while preserving the exact spectrum (SSP), the ordered multiplicity list (SMP), or the rank (SAP). These matrices, which are necessary in order to state the Matrix Liberation Lemma, are defined Definition 7.1.

Definition 7.1. Let G be a graph and let $\overline{E}$ be the set of pairs $\{(i,j) : i < j, \{i,j\} \in E(\overline{G})\}$. Let $A \in \mathcal{S}(G)$ and $p = |\overline{E}|$.

1. The *SSP verification matrix* $\Psi_S(A)$ of A is the $p \times \binom{n}{2}$ matrix $\mathrm{TS_S}(A)[\overline{E}, :]$.
2. The *SMP verification matrix* $\Psi_M(A)$ of A is the $p \times \left(\binom{n}{2} + q\right)$ matrix $\mathrm{TS_M}(A)[\overline{E}, :]$.
3. The *SAP verification matrix* $\Psi_A(A)$ of A is the $p \times n^2$ matrix $\mathrm{TS_A}(A)[\overline{E}, :]$.

It was shown in [8] that A satisfies the SSP, SMP, or SAP, respectively, if and only if the matrix $\Psi_S(A)$, $\Psi_M(A)$, or $\Psi_A(A)$ has full rank p. Notice that in $\Psi_M(A)$ the columns $\operatorname{vec}_{\overline{E}}(A^0)$ and $\operatorname{vec}_{\overline{E}}(A^1)$ are always zero, so we may omit them for verifying if A has the SMP or not.

The *support* of a vector $\mathbf{x}$ is the set of indices of nonzero coordinates of $\mathbf{x}$, and is denoted by $\operatorname{supp}(\mathbf{x})$. The **next result** is a special case of [19, Corollary 2.2] considering the intersection of only two manifolds. #16 (1006-LH)

Lemma 7.2. [19, Corollary 2.2] *Assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect transversally at $\mathbf{y}$, and let $\mathbf{v}$ be a common tangent to each of $\mathcal{M}_1$ and $\mathcal{M}_2$ with $||\mathbf{v}|| = 1$. Then for every $\epsilon > 0$ there exists a point $\mathbf{y}' \neq \mathbf{y}$ such that $\mathcal{M}_1$ and $\mathcal{M}_2$ intersect transversally at $\mathbf{y}'$, and*

$$\left\| \frac{1}{\|\mathbf{y} - \mathbf{y}'\|}(\mathbf{y} - \mathbf{y}') - \mathbf{v} \right\| < \epsilon.$$

Lemma 7.3 (Matrix Liberation Lemma)**.** *Let G be a graph and $A \in \mathcal{S}(G)$. Let Ψ be one of the following:*

- *Case 1.* $\Psi = \Psi_S(A)$.
- *Case 2.* $\Psi = \Psi_M(A)$.

- *Case 3.* $\Psi = \Psi_A(A)$.

Suppose $\mathbf{x}$ *is a vector in the column space of* Ψ *such that the complement of* $\operatorname{supp}(\mathbf{x})$ *corresponds to a linearly independent set of rows in* Ψ*. Let* H *be a spanning subgraph of* $\overline{G}$ *whose edges correspond to* $\operatorname{supp}(\mathbf{x})$*. Then* A *can be perturbed to* $A' \in \mathcal{S}(G \cup H)$ *such that:*

- *Case 1.* A' *satisfies the SSP with the same spectrum as* A*.*
- *Case 2.* A' *satisfies the SMP with the same ordered multiplicity list as* A*.*
- *Case 3.* A' *satisfies the SAP with the same rank as* A*.*

Proof. We prove the result in Case 1, as the other cases follow by similar arguments. Assume that $\Psi = \Psi_S(A)$. Let $\overline{E} = \{(i,j) : i < j, \{i,j\} \in E(\overline{G})\}$. Since $\mathbf{x}$ is in the column space of Ψ and the column space of $\mathrm{TS}_\mathrm{S}(A)$ is the tangent space of the isospectral manifold $\mathcal{E}_A$ at A, there is a matrix B in the tangent space such that $\operatorname{vec}_{\overline{E}}(B) = \mathbf{x}$. We may scale B such that $\|B\| = 1$.

Let V be the subspace (which is a manifold) of $n \times n$ symmetric matrices whose i,j-entry is zero if $\{i,j\} \in E(\overline{G \cup H})$. Then $V^\perp$ contains matrices whose nonzero entries appear only at those pairs that correspond to $E(\overline{G \cup H})$, which is also the complement of $\operatorname{supp}(\mathbf{x})$. By our assumption, the set of rows in $\mathrm{TS}_\mathrm{S}(A)$ corresponding to the complement of $\operatorname{supp}(\mathbf{x})$ is linearly independent. By Remark 6.2, $\mathcal{E}_A$ and V intersect transversally at A. Also, B is a common tangent to V and $\mathcal{E}_A$.

Applying Lemma 7.2 with two manifolds V and $\mathcal{E}_A$, $\mathbf{y} = A$, and $\mathbf{v} = B$, for every $\epsilon > 0$ there is a matrix A' with

$$\left\| \frac{1}{\|A - A'\|}(A - A') - B \right\| < \epsilon.$$

such that $\mathcal{E}_A$ and V intersect transversally at A'.

We may pick ϵ small enough such that the nonzero entries of B do not vanish in $\frac{1}{\|A-A'\|}(A - A')$, so the entries of A' corresponding to $E(H)$ are nonzero. Since A' is close to A, the nonzero entries of A do not vanish in A', so entries of A' corresponding to $E(G)$ are nonzero. All these facts and $A' \in V$ imply $A' \in \mathcal{S}(G \cup H)$. This means $\mathcal{E}_A$ and $\mathcal{S}(G \cup H)$ intersect transversally at A', so A' has the SSP and $\operatorname{spec}(A') = \operatorname{spec}(A)$. ☐

Remark 7.4. A vector $\mathbf{x}$ as in the Matrix Liberation Lemma exists if and only if no row of Ψ is zero. In this case, a vector $\mathbf{x}$ with every entry nonzero exists in the column span of Ψ, the graph $H = \overline{G}$, and Lemma 7.3 guarantees a matrix with the same spectrum as A such that all off-diagonal entries are nonzero. For the SSP, the condition that no row of $\Psi_S(A)$ is zero is equivalent to A not having a pair of rows i and j such that $a_{ik} = 0$ for $k \neq i$, $a_{jk} = 0$ for $k \neq j$, and $a_{ii} = a_{jj}$. #17

We apply the Matrix Liberation Lemma to prove additional results.

Lemma 7.5 (Augmentation Lemma)**.** *Let* G *be a graph on vertices* $\{1, \ldots, n\}$ *and* $A \in \mathcal{S}(G)$*. Suppose* A *has the SSP and* λ *is an eigenvalue of* A *with multiplicity* $k \geq 1$*. Suppose that* α *is a subset of* $\{1, \ldots, n\}$ *of cardinality* $k + 1$ *with the property that for every eigenvector* $\mathbf{x}$ *of* A *corresponding to* λ*,* $|\operatorname{supp}(\mathbf{x}) \cap \alpha| \geq 2$*. Construct* H *from* G *by appending vertex* $n + 1$ *adjacent exactly to the vertices in* α*. Then there exists a matrix* $A' \in \mathcal{S}(H)$ *such that* A' *has the SSP, the multiplicity of* λ *has increased from* k *to* $k + 1$*, and other eigenvalues and their multiplicities are unchanged from those of* A*.*

Proof. Consider $A_\lambda = A \oplus [\lambda]$. By Lemma 6.3, $\Psi_S(A_\lambda)$ has the form $\Psi_S(A) \oplus (A - \lambda I)$ after suitable permutation of rows and columns. The rows of $A - \lambda I$ with index not in α (equivalently, the rows of $\Psi_S(A_\lambda)$ indexed by $(j, n+1)$ with $j \notin \alpha$) are linearly independent; otherwise there is an eigenvector of A corresponding to λ whose support is disjoint from α. From this, the block diagonal structure of $\Psi_S(A_\lambda)$, and the fact that A has the SSP, all the rows of $\Psi_S(A_\lambda)$ associated with non-edges of H are linearly independent.

Let N be **an** $n \times k$ matrix whose columns are a basis for the null space of $A - \lambda I$. We show that every order k submatrix of $N[\alpha, :]$ is invertible: Suppose some $k \times k$ submatrix M of $N[\alpha, :]$ is not invertible. Then there is a vector $\mathbf{z}$ such that $M\mathbf{z} = \mathbf{0}$. But then $N\mathbf{z}$ is an eigenvector of $A - \lambda I$ whose support intersects α in at most one element, which contradicts the assumptions. #18

Since the rows of the $(k+1) \times k$ matrix $N[\alpha, :]$ are linearly dependent, and each of the $k \times k$ submatrices of $N[\alpha, :]$ is invertible, each coefficient in a dependence relation is nonzero. This gives a vector $\mathbf{y}$ with $\mathbf{y}^\top N = \mathbf{0}$ and $\operatorname{supp}(\mathbf{y}) = \alpha$. Since $\mathbf{y}$ is orthogonal to each column of N, $\mathbf{y}$ is in the row space (which equals the column space) of $A - \lambda I$. Since $\Psi_S(A_\lambda)$ has the form $\Psi_S(A) \oplus (A - \lambda I)$, there is a vector $\hat{\mathbf{y}}$ in the column space of $\Psi_S(A)$ such that $\operatorname{supp}(\hat{\mathbf{y}})$ is those edges $\{j, n+1\}$ with $j \in \alpha$. The result now follows from Lemma 7.3. $\square$

Corollary 7.6. *For any list of distinct real numbers $\lambda_1 < \cdots < \lambda_{n-1}$ and integer k with $1 \le k \le n-1$, there exists a matrix $A \in \mathcal{S}(C_n)$ with the SSP such that A has eigenvalues λ_k with $\operatorname{mult}_A(\lambda_k) = 2$ and $\operatorname{mult}_A(\lambda_i) = 1$ for $i \ne k$.*

Proof. By [8, Remark 15], there is a matrix $A \in \mathcal{S}(P_{n-1})$ with the SSP and $\operatorname{spec}(A) = \{\lambda_1, \ldots, \lambda_{n-1}\}$; let $\mathbf{v} = [v_i]$ be an eigenvector of A with respect to λ_k. By the structure of A and $(A - \lambda I)\mathbf{v} = \mathbf{0}$, if $v_1 = 0$ then we can see inductively that $v_2, \ldots, v_n$ are all zero, so $v_1 \ne 0$; similarly $v_{n-1} \ne 0$. Then by applying Lemma 7.5 with $\alpha = \{1, n-1\}$, there is a matrix $A' \in \mathcal{S}(C_n)$ with the SSP and $\operatorname{spec}(A') = \{\lambda_1, \ldots, \lambda_{k-1}, \lambda_k, \lambda_k, \lambda_{k+1}, \ldots, \lambda_{n-1}\}$. $\square$

The next result is not required for the rest of the paper, but it gives a different way to compute the verification matrices. This result displays the verification matrices as the coefficient matrices of systems of homogeneous equations with the variables on the left (for the traditional view transpose the verification matrices). Let $X_{ij} = E_{ij} + E_{ji}$ and $E_o = \{(i,j) : 1 \le i < j \le n\}$ the set of off-diagonal pairs.

Proposition 7.7. *Let G be a graph and $A \in \mathcal{S}(G)$. Then* ***the following statements are true.*** #19

1. *The (i,j)-row of $\Psi_S(A)$ is $\operatorname{vec}_{E_o}(AX_{ij} - X_{ij}A)$.*
2. *The SMP verification matrix $\Psi_M(A)$ is* ***formed*** *by augmenting $\Psi_S(A)$ with q columns $\operatorname{vec}_{\overline{E}}(A^k)$ for $k = 0, 1, \ldots, q-1$, where $\overline{E} = \{(i,j) : i < j, \{i,j\} \in E(\overline{G})\}$.* #20 (1006)
3. *The (i,j)-row of $\Psi_A(A)$ is $\operatorname{vec}_{E_o}(AX_{ij})$.*

Proof. Let n be the number of vertices of G. For fixed i, j with $1 \le i \le j \le n$, the (i,j)-row of $\Psi_S(A)$ is the (i,j)-row of $\mathrm{TS}_{\mathrm{S}}(A)$ by definition, and the (k, ℓ)-entry is

$$\begin{aligned}
\mathbf{e}_i^\top(AK_{k\ell} + K_{\ell k}A)\mathbf{e}_j &= \mathbf{e}_i^\top(AE_{k\ell} - AE_{\ell k} + E_{\ell k}A - E_{k\ell}A)\mathbf{e}_j \\
&= \mathbf{e}_i^\top(A\mathbf{e}_k\mathbf{e}_\ell^\top - A\mathbf{e}_\ell\mathbf{e}_k^\top + \mathbf{e}_\ell\mathbf{e}_k^\top A - \mathbf{e}_k\mathbf{e}_\ell^\top A)\mathbf{e}_j \\
&= \mathbf{e}_i^\top A\mathbf{e}_k\mathbf{e}_\ell^\top\mathbf{e}_j - \mathbf{e}_i^\top A\mathbf{e}_\ell\mathbf{e}_k^\top\mathbf{e}_j + \mathbf{e}_i^\top\mathbf{e}_\ell\mathbf{e}_k^\top A\mathbf{e}_j - \mathbf{e}_i^\top\mathbf{e}_k\mathbf{e}_\ell^\top A\mathbf{e}_j \\
&= \mathbf{e}_\ell^\top\mathbf{e}_j\mathbf{e}_i^\top A\mathbf{e}_k - \mathbf{e}_k^\top\mathbf{e}_j\mathbf{e}_i^\top A\mathbf{e}_\ell + \mathbf{e}_k^\top A\mathbf{e}_j\mathbf{e}_i^\top\mathbf{e}_\ell - \mathbf{e}_\ell^\top A\mathbf{e}_j\mathbf{e}_i^\top\mathbf{e}_k \\
&= \mathbf{e}_k^\top A\mathbf{e}_i\mathbf{e}_j^\top\mathbf{e}_\ell - \mathbf{e}_k^\top\mathbf{e}_j\mathbf{e}_i^\top A\mathbf{e}_\ell + \mathbf{e}_k^\top A\mathbf{e}_j\mathbf{e}_i^\top\mathbf{e}_\ell - \mathbf{e}_k^\top\mathbf{e}_i\mathbf{e}_j^\top A\mathbf{e}_\ell \\
&= \mathbf{e}_k^\top(AE_{ij} - E_{ji}A + AE_{ji} - E_{ij}A)\mathbf{e}_\ell \\
&= \mathbf{e}_k^\top(AX_{ij} - X_{ij}A)\mathbf{e}_\ell,
\end{aligned}$$

which is the (k,ℓ)-entry of $AX_{ij} - X_{ij}A$. This deals with **Statement** (1). For **Statement** (2), #21
it follows directly from Definition 6.1 and Definition 7.1. Finally, for the (i,j)-row of $\Psi_A(A)$, the (k,ℓ)-entry is

$$\begin{aligned}
\mathbf{e}_i^\top(AE_{k\ell} + E_{\ell k}A)\mathbf{e}_j &= \mathbf{e}_i^\top A\mathbf{e}_k\mathbf{e}_\ell^\top\mathbf{e}_j + \mathbf{e}_i^\top\mathbf{e}_\ell\mathbf{e}_k^\top A\mathbf{e}_j \\
&= \mathbf{e}_\ell^\top\mathbf{e}_j\mathbf{e}_i^\top A\mathbf{e}_k + \mathbf{e}_k^\top A\mathbf{e}_j\mathbf{e}_i^\top\mathbf{e}_\ell \\
&= \mathbf{e}_k^\top A\mathbf{e}_i\mathbf{e}_j^\top\mathbf{e}_\ell + \mathbf{e}_k^\top A\mathbf{e}_j\mathbf{e}_i^\top\mathbf{e}_\ell \\
&= \mathbf{e}_k^\top(AX_{ij})\mathbf{e}_\ell,
\end{aligned}$$

which is the (k,ℓ)-entry of AX_{ij}. This gives (3). □

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