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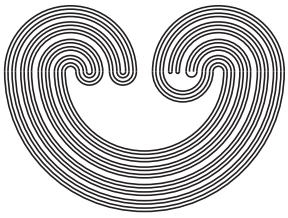

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by

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SOME ASPECTS OF TOPOLOGICAL ALGEBRA AND REMAINDERS OF TOPOLOGICAL GROUPS

A.V. ARHANGEL'SKII

Abstract. In the Introduction a very brief survey of some classical results and ideas of topological algebra is given. The principal interest in the article is directed at remainders of topological groups; here some new results are obtained. Thus, we continue the research line adopted in [8], [9], [10], [11]. Several results from these articles are improved. It is established that if a remainder of a non-locally compact topological group G is the union of a finite collection of metrizable spaces, then G is metrizable. A far reaching generalization of this result is also given; it is based on the notion of a D-space. If X is an uncountable Tychonoff space, and bY is a Hausdorff compactification of the space $Y = C_p(X)$ such that the remainder $bY \setminus Y$ is homogeneous, then bY can be mapped continuously onto the Tychonoff cube I^{ω_1}. Some further results and open problems on remainders of topological groups are provided.

1. Introduction

Topological algebra and its objects may be of special interest to general topologists for several reasons. We mention below just two of them. First, the presence of an algebraic structure nicely related to a topology changes dramatically relationship between topological invariants. Important classical results in this direction

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are well known; enough to mention Birkhgoff-Kakutani's theorem that first countability is equivalent to metrizability in topological groups, Pontryagin's theorem that every topological group is a Tychonoff space, and Bourbaki's theorem that every locally compact topological group is paracompact (see for a discussion [26], [5], and [18]). We present below some recent results of this kind. Recall that a *paratopological group* is a group with a Hausdorff topology such that the multiplication is jointly continuous. First countable paratopological groups, unlike first countable topological groups, needn't be metrizable. However, it turns out that the existence of G_δ-points in paratopological groups is intimately related to submetrizability [12]. Some partial results related to the following major open problem are given below.

Problem 1.1. [12] *Is every regular first countable paratopological group submetrizable?*

Pontryagin's theorem also does not generalize to paratopological groups. However, the next question is open.

Problem 1.2. *Is every regular Hausdorff paratopological group Tychonoff?*

The second reason for a general topologist to be interested in topological algebra lies in the following. The structures of topological algebra naturally produce some standard constructions unavailable in pure general topology. These constructions lead to algebraico-topological objects which turn out to be highly nontrivial when treated as topological spaces. An important construction of this kind is that of the free topological group $F(X)$ over a Tychonoff space X.

We may also apply classical constructions of general topology (like that of the Stone-Čech compactification) to various concrete objects of topological algebra, which further expands the range of topological spaces within our reach.

In particular, we are going to consider the remainders of topological groups in Hausdorff compactifications. Of course, the spaces, "generated" by topological groups in this way, do not inherit from them any natural algebraic structure.

By a *remainder* of a Tychonoff space X we understand the subspace $bX \setminus X$ of a Hausdorff compactification bX of X.

A famous classical result on duality between properties of spaces and properties of their remainders is the following theorem of M. Henriksen and J. Isbell [21]:

Theorem 1.3. *A Tychonoff space X is of countable type if and only if the remainder in any (in some) Hausdorff compactification of X is Lindelöf.*

A space X is of *countable type* if every compact subspace P of X is contained in a compact subspace $F \subset X$ with a countable base of open neighbourhoods in X. All metrizable spaces and all locally compact Hausdorff spaces, as well as all Čech-complete spaces, are of countable type [1]. It follows from the theorem of Henriksen and Isbell that every remainder of a metrizable space is Lindelöf and hence, paracompact.

If *every* remainder of a Tychonoff space X has a certain property $\mathcal{P}$, then we say, following [21], that X *has property $\mathcal{P}$ at infinity*.

Recall that *paracompact p-spaces* [1] are preimages of metrizable spaces under perfect mappings. A mapping is said to be *perfect* if it is continuous, closed, and all fibers are compact. *A Lindelöf p-space* is a preimage of a separable metrizable space under a perfect mapping [1].

For the definition of a p-space see [1], where this notion was introduced. It was shown in [1] that every p-space is of countable type, and that every metrizable space is a p-space. It was shown by B.A. Pasynkov that if a topological group G is of countable type then G is a paracompact p-space; even more, there exists in this case a compact subgroup H of G such that the quotient space G/H is metrizable. Pasynkov called such groups G *almost metrizable* (see [26]).

Clearly, every separable metrizable space has a separable metrizable remainder. Here is a parallel result from [8]:

Theorem 1.4. *If X is a Lindelöf p-space, then any remainder of X is a Lindelöf p-space.*

Some of the general remarks made above are illustrated by the following example.

Example 1.5. Let Seq be the simplest infinite compact space, that is, the Alexandroff one-point compactification of the discrete space of natural numbers. Consider the free topological group $F(Seq)$ of

the space Seq, and let Z be the Stone-Čech remainder of the space $F(Seq)$. Then Z has the following curious collection of properties:

1) Z is nowhere locally compact;

2) Z is Čech-complete, and the Souslin number of Z is countable. Moreover, ω_1 is a precaliber of Z;

3) Z is not paracompact;

4) The closure in Z of every σ-compact subset of Z is compact; hence, Z is not separable. However, Z contains a dense Lindelöf Čech-complete subspace. A similar example was considered in [10], where the proofs of the above statements can be found and some further properties of Z are mentioned.

This example shows that the Stone-Čech remainder of a quite nice, countable and sequential, topological group may be a space of a very different kind, with an unusual combination of properties. By the way, we do not know if Z is homogeneous, or whether Z is normal.

2. Remainders that are unions, and metrizability of groups

We now embark on a more detailed discussion of remainders of topological groups. It turns out, not unexpectedly, that these objects are much more sensitive to properties of their remainders than topological spaces are in general. However, properties of remainders of topological groups are not yet well understood. In this direction we mention the following recent results:

Theorem 2.1. [8], [9] *Suppose that G is a topological group, and that M is a remainder of G in a Hausdorff compactification bG. Then the following conditions are equivalent:*

a) M has a G_δ-diagonal;

b) M is submetrizable;

c) M is metrizable;

d) M has a point-countable base.

Besides, if G is not locally compact, then each of these conditions implies that G, Y, and bG are separable and metrizable.

Here is a partial strengthening of the above theorem:

Theorem 2.2. *If a non-locally compact topological group G has a remainder which is the union of a finite collection of metrizable subspaces, then G is metrizable.*

Proof. Let bG be a Hausdorff compactification of G and $\eta = \{M_i : i = 1, ..., l\}$ be a finite collection of metrizable subspaces of bG such that $bG \setminus G = \cup\eta$.

Claim 1: There exist a non-empty open subset U of bG and a set $A \subset bG$ such that $U \subset \overline{A}$, and bG is first countable at each $x \in A$.

Indeed, we can easily find, by a standard argument, a non-empty open subset U of bG and an element M_k of η such that U is contained in the closure of M_k. Put $A = U \cap M_k$. The subspace A is first countable, since $A \subset M_k$ and M_k is metrizable. However, A is, obviously, dense in U. It follows that U is first countable at each point of A. Since U is open in bG, we conclude that bG is first-countable at each $x \in A$. Thus, Claim 1 is verified.

For a subset B of bG, we denote by $[B]_\omega$ the set

$$\cup\{\overline{P} : P \subset B, |P| \leq \omega\}.$$

Claim 2: If $A \subset bG$ and the closure of A intersects G, then the set $[A]_\omega$ also intersects G.

Indeed, otherwise $[A]_\omega \subset (bG \setminus G)$ which implies that $[A]_\omega$ is the union of a finite collection of metrizable spaces. Since $[A]_\omega$ is, obviously, countably compact, it follows that $[A]_\omega$ is compact. Indeed, by a theorem of A. Ostaszewski [24], if a countably compact Tychonoff space Y is the union of a countable family of metrizable spaces, then Y is compact (A. Ostaszewski, see for a discussion and generalizations [13] and [19]). Therefore, $[A]_\omega = \overline{A}$ and $G \cap \overline{A} = \varnothing$, a contradiction. Thus, Claim 2 is also established.

Let us fix the sets A and U such as in Claim 1. Then the closure of A intersects G, so that we can apply Claim 2. It follows that there exist a countable subset C of A and a point $b \in G$ such that $b \in \overline{C}$. Observe that bG is first countable at each point of C, since C is a subset of A. It follows that the space bG has a countable π-base at b (the union of countable bases at the points of C). Since G is dense in bG, we conclude that G has a countable π-base at b. But G is a topological group. Hence, G is first countable and metrizable. □

In the last theorem it is enough to assume that the remainder is the union of a finite collection of subspaces with a point-countable base.

Theorem 2.1 plays an important role in the proof of the following theorem:

Theorem 2.3. *Suppose that G is a non-locally compact paratopological group, and that bG is a Hausdorff compactification of G. Suppose further that the remainder $Z = bG \setminus G$ is the union of a countable family of metrizable compacta. Then G, Z, and the compactification bG are separable and metrizable.*

Proof. It follows from the assumption about the remainder Z that the space G is Čech-complete. Therefore, by a result of A. Bouziad and E. Reznichenko [15], [25], G is a topological group. The remainder Z has a G_δ-diagonal, since Z, obviously, has a countable network. Now it follows from Theorem 2.1 that both spaces G and Z are separable and metrizable. Therefore, by the addition theorem for the weight in compacta [18], the compactification bG is separable and metrizable as well. □

We now present some far reaching generalizations of Theorem 2.2. We start with a simple general statement. Recall that a space X is *ω-bounded* if the closure of every countable subset of X is compact.

Proposition 2.4. *Suppose that G is a non-locally compact topological group, and that bG is a Hausdorff compactification of G such that the remainder $Y = bG \setminus G$ is of countable π-character at a dense set of points and satisfies the condition: every ω-bounded subspace of Y is compact. Then G is metrizable.*

Proof. Let A be the set of points of Y at which the π-character of Y is countable. By the assumption, A is dense in Y. Let P be the union of the closures in bG of all countable subsets of A. Clearly, the subspace P is ω-bounded.

Case 1: $P \subset Y$. Then P is compact, by the assumption. Observe that P is dense in Y, since P contains A. It follows that $P = Y$. Hence Y is compact and therefore, G is locally compact, a contradiction. Thus, Case 1 is impossible.

Case 2: $G \cap P \neq \varnothing$. Then we can find a point $b \in G$ and a countable set $M \subset A$ such that $b \in \overline{M}$. Observe that Y is dense in bG, since G is nowhere locally compact. It follows that bG has a countable π-base η_a at each point a of M. Then $\eta = \cup\{\eta_a : a \in M\}$ is a countable π-base of bG at the point b. However, G is dense in bG, since G is nowhere locally compact. Hence, $\xi = \{U \cap G : U \in \eta\}$ is a countable π-base of the space G at b. Since G is a topological group, it follows that G is metrizable [3]. □

Some results obtained with the help of Proposition 2.4 are based on the notion of a D-space introduced by E. van Douwen. Recall that a *neighbourhood assignment* on a topological space X is a mapping p of X into the topology of the space X such that $x \in p(x)$, for each $x \in X$. Thus, $p(x)$ is an open neighbourhood of x. A space X is said to be a *D-space* if for every neighbourhood assignment p on X there exists a closed discrete subset A of X such that the family $p(A) = \{p(x) : x \in A\}$ covers X. The class of D-spaces includes the classes of metrizable spaces, Moore spaces, spaces with a σ-discrete network (for definitions and comments see [7] and [20]).

We will use below the following result from [19]:

Theorem 2.5. *If a Tychonoff countably compact space X is the union of a countable family of D-spaces, then X is compact.*

We also need the next obvious lemma:

Lemma 2.6. *Let $\mathcal{E}$ be the class of all topological spaces X such that the π-character of X is countable at a dense set of points. Then every Tychonoff space Y which is the union of a finite family of spaces belonging to $\mathcal{E}$ also belongs to $\mathcal{E}$.*

We use the symbol $\mathcal{E}$ in the same sense in Lemma 3.4 below.

Now we can easily establish our main general results:

Theorem 2.7. *Suppose that G is a non-locally compact topological group, and that the remainder of G in a Hausdorff compactification bG is the union of a finite collection of hereditarily D-spaces each of which is first countable (of countable π-character) at a dense set of points. Then G is metrizable.*

Proof. It follows from Theorem 2.5 that every ω-bounded subspace of the remainder $Y = bG \setminus G$ is compact. By Lemma 3.4, the π-character of Y is countable at a dense set of points. Now Proposition 2.4 implies that G is metrizable. □

Corollary 2.8. *Suppose that G is a non-locally compact topological group such that the remainder of G in some Hausdorff compactification bG of G is the union of a finite collection of subspaces each of which either is first countable and has a σ-discrete network, or has a point-countable base. Then G is metrizable.*

Proof. In view of Theorem 2.7, it is enough to mention that all Tychonoff spaces with a point-countable base are hereditarily D-spaces [13], and that each Tychonoff space with a σ-discrete network is also a hereditarily D-space (see for comments [7]). □

Here is another application of Proposition 2.4:

Theorem 2.9. *Suppose that G is a topological group with a Hausdorff compactification bG such that every ω-bounded subspace of the remainder $Y = bG \setminus G$ is compact. Then at least one of the following two conditions holds:*

a) bG can be continuously mapped onto the Tychonoff cube I^{ω_1};

b) G is metrizable.

Proof. If G is not metrizable and G is locally compact, then G contains a topological copy of D^{ω_1} (see [18]). Since the space bG is normal, it follows that bG can be continuously mapped onto the Tychonoff cube I^{ω_1}.

So it remains to consider the case when G is not locally compact. Then both G and Y are dense in bG. Assume now that condition a) doesn't hold. Then, by Shapirovskij's Theorem in [27] (see also [23]), the set A of all points $x \in bG$ such that the π-character of bG at x is countable is dense in bG. Since G is dense in bG, it follows that the π-character of G is countable at every point of $A \cap G$. Thus, if $A \cap G$ is non-empty, then G is metrizable.

It remains to consider the case when $A \cap G = \varnothing$. Then, clearly, $A \subset Y = bG \setminus G$. Observe, that for each countable subset C of A, the closure of C in bG is contained in A. Hence, A is ω-bounded. Therefore, by the assumption, A is compact. Then $A = bG$, since A is dense in bG. Hence, $A \cap G \neq \varnothing$, a contradiction. □

Theorem 2.10. *Suppose that G is a topological group with a Hausdorff compactification bG such that the remainder $Y = bG \setminus G$ is the union of a countable family γ of subspaces satisfying at least one of the following conditions:*

a) Each $X \in \gamma$ is hereditarily θ-refinable;

b) Each $X \in \gamma$ is a hereditarily D-space.

Then either G is metrizable, or bG can be continuously mapped onto the Tychonoff cube I^{ω_1}.

Proof. Assume that b) holds. Then, by Theorem 2.9, it is enough to refer to Theorem 2.5. If a) holds, we have to refer to a similar result on the union of θ-refinable spaces (see [6], [7]). □

3. Remainders with the Baire property

Recall that a space X has the *Baire property* if the intersection of every countable family of open dense subsets of X is dense in X.

First, we mention below several conditions that guarantee that a topological group has a remainder with the Baire property. After that we establish some properties of topological groups that have a remainder with the Baire property.

The next fact is easy to establish:

Proposition 3.1. *If a topological group G is either pseudocompact or σ-compact, then every remainder of G in a Hausdorff compactification bG has the Baire property.*

Proof. If G is σ-compact, then each remainder of G is Čech-complete and therefore, has the Baire property.

Assume now that G is pseudocompact. The Stone-Čech compactification βG is a compact topological group containing G as a subgroup [16]. Fix $a \in \beta G \setminus G$, and put $Y = aG$. Clearly, Y is homeomorphic to G and is dense in βG. Hence, Y and the remainder $H = (\beta G) \setminus G$ are pseudocompact. It follows that the remainder $bG \setminus G$ is also pseudocompact and hence, has the Baire property. □

A very general sufficient condition for an arbitrary remainder of a topological group to have the Baire property can be easily derived from the following basic theorem on remainders of topological groups recently established in [11]:

Theorem 3.2. *For every topological group G, either every remainder of G is pseudocompact, or G is a paracompact p-space (and then every remainder of G is Lindelöf).*

Corollary 3.3. *If a topological group G is not a paracompact p-space, then every remainder of G in a Hausdorff compactification has the Baire property.*

Proof. This statement follows from Theorem 3.2, since each pseudocompact space has the Baire property. □

For the spaces with the Baire property we have the following version of Lemma 2.6, the proof of which is also obvious. Recall that $\mathcal{E}$ is the class of all spaces the π-character of which is countable at a dense set of points.

Lemma 3.4. *Every Tychonoff space Y with the Baire property which is the union of a countable family of spaces belonging to $\mathcal{E}$ also belongs to $\mathcal{E}$.*

The next theorem complements well some results of the preceding section.

Theorem 3.5. *Suppose that G is a non-locally compact topological group, and that the remainder of G in some Hausdorff compactification bG of G has the Baire property. Suppose also that the remainder $Y = bG \setminus G$ is the union of a countable family of hereditarily D-spaces each of which is of countable π-character at a dense set of points. Then G is metrizable.*

Proof. The proof of Theorem 3.5 is almost the same as the proof of Theorem 2.7; instead of Lemma 2.6 we apply Lemma 3.4. □

Corollary 3.6. *If some remainder of a topological group G in a Hausdorff compactification of G is the union of a countable family of Moore spaces, then G is a paracompact p-space.*

Proof. Assume the contrary. Then every remainder of G in a Hausdorff compactification has the Baire property, by Corollary 3.3. Observe that every Moore space is a first countable hereditarily D-space (see [7]). Therefore, by Theorem 3.5, G is metrizable. Hence G is a paracompact p-space [1], a contradiction. □

A similar argument shows that the next statement holds.

Corollary 3.7. *If some remainder of a topological group G in a Hausdorff compactification of G is the union of a countable family of spaces with a point-countable base, then G is a paracompact p-space.*

Each of the last two statements implies the next result:

Corollary 3.8. *If some remainder of a topological group G is the union of a countable family of metrizable spaces, then G is a paracompact p-space.*

We now repeat an open problem from [10]:

Problem 3.9. *Is every non-locally compact topological group with a first countable remainder metrizable?*

Most of the results on topological groups in this section do not generalize to paratopological groups. This is demonstrated by an example from [8]: Sorgenfrey line, which is a non-metrizable paratopological group, has a remainder homeomorphic to Sorgenfrey line which is submetrizable.

Problem 3.10. *Suppose that G is a Tychonoff paratopological group with a dense Čech-complete subspace. Must G be a p-space? Must G be a topological group?*

4. Homogeneity restrictions on remainders of groups

Results of this section show, maybe somewhat unexpectedly, that homogeneity conditions play a role in addition theorems for compacta. Note that a stronger version of homogeneity is the requirement for a space to be a semitopological group. Recall that a topological space with a group operation is said to be a *semitopological group* if the multiplication is separately continuous.

Theorem 4.1. *Suppose that $B = G \cup Y$, where B is a compact Hausdorff space, G is a metrizable non-locally compact topological group dense in B, and Y is a semitopological group. Then B is separable and metrizable.*

Proof. Clearly, Y is also dense in B. If G is countably compact, then G is compact, since G is metrizable. Hence, G is closed in B in this case, and therefore, $G = B$. Hence, B is separable and metrizable.

Assume now that G is not countably compact. Then there exist a countable set $A \subset G$ and a point $b \in B \setminus G$ such that $b \in \overline{A}$. Notice that $b \in Y$, and that B is first countable at each point of A, since G is dense in B. Therefore, B has a countable π-base at b.

Since Y is dense in B, it follows that Y has a countable π-base at b. However, every Tychonoff semitopological group of countable π-character has a G_δ-diagonal (see [12]). Hence, Y has a G_δ-diagonal. Since this property is inherited by subspaces, the remainder $B \setminus G$ of G in B also has a G_δ-diagonal. By Theorem 2.1, both G and $B \setminus G$ are separable and metrizable. Hence, by the addition theorem (see [18]), B is separable and metrizable. □

Here is a version of the preceding result in which the restriction on G is weakened:

Theorem 4.2. *Suppose that G is a non-locally compact topological group in which points are G_δ-sets, and that bG is a Hausdorff compactification of G such that the remainder $Y = bG \setminus G$ is a paratopological group. Then G is separable and metrizable.*

Proof. The restriction on G implies that G is submetrizable, since G is a topological group [3]. By Theorem 3.2, either G is a paracompact p-space, or the remainder Y is pseudocompact.

Case 1: G is a paracompact p-space; then G is metrizable, since every submetrizable paracompact p-space is metrizable [1]. It follows from Theorem 4.1 that G is separable.

Case 2: Y is pseudocompact. Then Y is a topological group, since every pseudocompact paratopological group is a topological group (see [14] and [5]). Clearly, Y is dense in bG. Thus, G is a remainder of Y. Since Y is a topological group and G has a G_δ-diagonal, it follows from Theorem 2.1 that Y and G are separable and metrizable. □

Theorem 4.3. *Suppose that X is a compact Hausdorff space such that X cannot be mapped continuously onto I^{ω_1}. Suppose also that $X = Y \cup Z$, where none of the spaces Y and Z is locally compact, Y is homeomorphic to a topological group, and Z is homeomorphic to a semitopological group. Then X is separable and metrizable.*

Proof. Notice that our assumptions about Y and Z obviously imply that Y and Z are dense in X.

The assumption that X cannot be mapped continuously onto I^{ω_1} implies that the π-character of X at some point a is countable [27]. If $a \in Y$, then the π-character of Y at a is also countable, since Y is dense in X. Then Y is metrizable, since Y is a topological group. Now it follows from Theorem 4.1 that X is separable and metrizable.

Let us consider the case when $a \in Z$. Then the π-character of Z at a is also countable, since Z is dense in X. It follows that Z has a G_δ-diagonal, since Z is homeomorphic to a semitopological group (see [12]). Hence, the remainder of Y in X (which may be smaller than Z) has a G_δ-diagonal as well. Therefore, X is metrizable and separable, by Theorem 2.1. □

Recall that if X is a Tychonoff space, then $C_p(X)$ stands for the space of continuous real-valued functions on X in the topology of pointwise convergence [4].

Theorem 4.4. *Suppose that X is a compact Hausdorff space that cannot be mapped continuously onto I^{ω_1}. Suppose also that $X = Y \cup Z$, where Y is dense in X, and Y is homeomorphic to $C_p(M)$, for some infinite Tychonoff space M, and Z is homogeneous. Then Y is separable and metrizable, and $|X| \leq 2^\omega$.*

Proof. We first note that the subspace $H = X \setminus Y$ is dense in X. Indeed, otherwise, Y would have been locally compact which is impossible, since M is infinite (see [4]). It follows that Z is dense in X.

By Shapirovskij Theorem [27] (see also [19]), the set T of points at which the space X has a countable π-base is dense in X. Either $Y \cap T \neq \varnothing$ or $Z \cap T \neq \varnothing$. Since both Y and Z are dense in X, it follows that either the space Y has a point of countable π-character (in itself), or Z has a point of countable π-character (in itself).

In the first case the space Y is metrizable, since it is a topological group. Then Y is separable and has a countable base $\mathcal{B}$, since the Souslin number of Y is countable [4]. Since Y is dense in X, and X is regular, it follows that X has a countable π-base $\mathcal{P}$. Since Z is dense in X, the family $\eta = \{U \cap Z : U \in \mathcal{P}\}$ is a countable π-base of Z.

Since Z is homogeneous and Hausdorff, we can apply a well known theorem of van Douwen [16] saying that $|Z| \leq 2^\omega$. We also have $|Y| \leq 2^\omega$, since Y is separable and metrizable. Hence $|X| \leq 2^\omega$.

In the second case, the π-character of X is countable at some point of Z. Since Z is dense in X, and Z is homogeneous, it follows that the π-character of Z is countable at every point. Recall that the remainder $H = X \setminus Y$ of Y is also dense in X. Hence, H is dense in Z. Therefore, the π-character of the remainder H is countable

at every point. Then, by a theorem in [10], the space $C_p(M)$ is separable and metrizable. The rest of the argument is the same as in case 1. □

We can drop the assumption that Y is dense in X if we add the requirement that Y and Z are disjoint.

Corollary 4.5. *Suppose that X is a compact Hausdorff space that doesn't admit a continuous mapping onto the Tychonoff cube I^{ω_1}, and that $X = Y \cup Z$, where Y is homeomorphic to $C_p(M)$, for some infinite Tychonoff space M, Z is homogeneous, and $Y \cap Z = \varnothing$. Then Y is separable and metrizable, and $|Z| \leq 2^{\omega}$.*

Proof. To reduce this statement to Theorem 4.4 it is enough to show that Y is dense in X. Assume the contrary. Then Z is locally compact at some point, since $X = Y \cup Z$ and X is compact. It follows that Z is locally compact, since Z is homogeneous. Put $B = \overline{Y}$. Clearly, $B \cap Z = B \setminus Y$, since Y and Z are disjoint. However, B is closed in X, and Z is locally compact. It follows that $B \setminus Y$ is a closed subspace of Z. Hence, $B \setminus Y$ is locally compact. Observe that $B \setminus Y$ is dense in B, since Y is nowhere locally compact. It follows that $B \setminus Y$ is open in B. But the sets $B \setminus Y$ and Y are disjoint. Therefore, Y is not dense in B, a contradiction. Hence, Y is dense in X, and we can refer to Theorem 4.4. □

Corollary 4.6. *If X is an uncountable Tychonoff space, and B is a Hausdorff compactification of the space $C_p(X)$ such that the remainder $Z = B \setminus C_p(X)$ is homogeneous, then the compactum B can be mapped continuously onto the Tychonoff cube I^{ω_1}.*

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