TOPOLOGY PROCEEDINGS

Volume 11, 1986

Pages 267–279

http://topology.auburn.edu/tp/

HE FUNDAMENTAL GROUP AND WEAKLY CONFLUENT MAPPING ON ANRs, I

by

Joachim Grispolakis

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings Department of Mathematics & Statistics Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

THE FUNDAMENTAL GROUP AND WEAKLY CONFLUENT MAPPING ON ANRs, I

Joachim Grispolakis (1)

1. Introduction

The purpose of this paper[2] is to study weakly confluent mappings defined on ANRs and on inverse limits of manifolds and to examine the affect of these mappings on the 1-homology groups and the fundamental groups of the spaces involved. It is proved that if $f: X \to Y$ is a weakly confluent mapping between compact, connected ANRs, and if every simple closed curve in Y is approximated by a spiral then $f_{\#}[H_1(X;\mathbb{Z})]$ has finite index in $H_1(Y;\mathbb{Z})$ (see Theorem 3.2). It is a natural question now to ask whether this relation between the first homology groups is sufficient for f to be homotopic to a weakly confluent mapping provided that X is a higher dimensional manifold. It is also proved (see Corollary 3.6) that if $f: X \to Y$ is a weakly confluent mapping from a compact connected n-manifold, $n \geq 3$, onto an ANR Y such that every simple closed curve is approximated by a spiral and $\pi(Y)$ is abelian, then f is homotopic to an open mapping from X onto Y.

It is of interest to know when weakly confluent mappings defined on manifolds can be approximated by confluent

---

[1]The author was partially supported by the National Research Foundation of Greece.

[2]This research was completed while the author was a visitor at the University of Saskatchewan, Canada.

mappings. It is shown that any n-dimensional compact Hausdorff space, $n \geq 3$, can be mapped by a weakly confluent mapping onto the wedge Y of finitely many compact connected ANRs with finite fundamental groups. On the other hand, for compact n-manifolds, $n \geq 3$, with finite fundamental group, there is not even a confluent map onto such ANRs (see [6]). It will be shown in [8] that even if the fundamental group of Y is a free product of finitely many finite groups, but Y is not a wedge, then there is no weakly confluent mapping from I^3 onto Y.

2. Preliminaries

By a *mapping* we mean a continuous function. A mapping $f: X \to Y$ from a compact Hausdorff space X onto a Hausdorff space Y is said to be *confluent* (respectively, *weakly confluent*) provided that for every compact connected subset K of Y and for each (resp., for some) component C of $f^{-1}(K)$ we have that $f(C) = K$. It is known that open mappings are confluent [17, p. 148].

By an n-*manifold* we mean a topological n-manifold with or without boundary, and by a Q-manifold we mean a Hilbert cube manifold [1]. PL means piecewise linear. By an ANR we mean a metric absolute neighbourhood retract. If a mapping f is non-homotopically trivial, then we write $f \text{ non} \simeq 0$. For a space X, $\check{H}^n(X;G)$, $\check{H}_n(X;G)$ and $H_n(X;G)$ denotes the n-Čech cohomology group, the n-Čech homology group and the n-singular homology group, respectively, of X with coefficient group G.

The following powerful results were obtained by J. J. Walsh in [13], [14] and [15].

2.1 *Theorem* (J. J. Walsh). *A mapping* $f: M \to Y$ *from a compact, connected* PL n-*manifold* M, $n \geq 3$, *into a compact connected* ANR Y *is homotopic to a monotone open (resp., to an open) mapping of* M *onto* Y *if and only if* $f_{\#}: \pi(M,x) \to \pi(Y,y)$ *is onto (resp.,* $f_{\#}\pi(M,x)$ *has a finite index in* $\pi(Y,y)$).

By using this result, it was proved in [6] that a mapping from a compact, connected PL n-manifold, $n \geq 3$, into a compact, connected ANR is homotopic to an open mapping if and only if it is homotopic to a confluent mapping.

3. Weakly confluent mappings of ANRs.

In [5] the following result was proved:

3.1 *Theorem. Let* $f: X \to Y$ *be a weakly confluent mapping from a compact Hausdorff space* X *onto a* PL n-*manifold,* $n \geq 2$, *or a* Q-*manifold. Then*

$$f^*: \check{H}^1(Y;\mathbb{Z}) \to \check{H}^1(X;\mathbb{Z})$$

is a monomorphism.

In fact, the theorem was proved for Y being an ANR where every simple closed curve can be approximated by a spiral in Y, that is, every simple closed curve S is the remainder of a compactification of a half-line L such that $L \cup S \cong \{(\rho,\theta): \rho = 1 \text{ or } \rho = \frac{2+e^{\theta}}{1+e^{\theta}} \text{ and } \theta \geq 0\}$. By using this theorem we shall give a necessary condition for a mapping to be homotopic to a weakly confluent mapping.

3.2 *Theorem.* *Let* $f: X \to Y$ *be a weakly confluent mapping from a compact connected* ANR X *onto an* ANR Y *such that every simple closed curve can be approximated by a spiral in* Y. *Then* $H_1(Y;\mathbb{Z})/f_*H_1(X;\mathbb{Z})$ *is a finite abelian group.*

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}(H_0(Y;\mathbb{Z}),\mathbb{Z}) & \to & H^1(Y;\mathbb{Z}) & \overset{\phi}{\to} & \mathrm{Hom}(H_1(Y;\mathbb{Z}),\mathbb{Z}) & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{Ext}(f_*,\mathrm{id})} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle \mathrm{Hom}(f_*,\mathrm{id})} & & \\ 0 & \to & \mathrm{Ext}(H_0(X;\mathbb{Z}),\mathbb{Z}) & \to & H^1(X;\mathbb{Z}) & \overset{\Psi}{\to} & \mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z}) & \to & 0 \end{array}$$

This diagram consists of two exact sequences (see [11, p. 243]) and a morphism between them, and all the subdiagrams commute. Since X and Y are connected, we have that

$$H_0(X;\mathbb{Z}) \cong H_0(Y;\mathbb{Z}) \cong \mathbb{Z}.$$

Hence, by taking the short free resolution $0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$ for $\mathbb{Z}$ we can see that $\mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = 0$. By the exactness of the two rows in the above diagram, we infer that ϕ and Ψ are isomorphisms. Since X and Y are compact ANRs, we have that $H^1(X;\mathbb{Z}) = \check{H}^1(X;\mathbb{Z})$ and $H^1(Y;\mathbb{Z}) = \check{H}^1(Y;\mathbb{Z})$. By Theorem 3.1, f^* is a monomorphism, and hence, $\mathrm{Hom}(f_*,\mathrm{id})$ is a monomorphism.

Suppose, now, on the contrary, that $H_1(Y;\mathbb{Z})/f_*H_1(X;\mathbb{Z})$ is an infinite abelian group. By West's result [16], Y has the homotopy type of a finite complex, and hence, $H_1(Y;\mathbb{Z})$ is a finitely presented abelian group. This implies that the group $G = H_1(Y;\mathbb{Z})/f_*H_1(X;\mathbb{Z})$ is a finitely presented infinite abelian group, and hence, we have that $G = \mathbb{Z}^m \times A$, where $m \geq 1$ and A is a finite abelian group (see [10, p. 49]). Let $p: H_1(Y;\mathbb{Z}) \to G$ be the natural projection, let

$i: G \to \mathbb{Z}^m \times A$ be the isomorphism, and let $j: \mathbb{Z}^m \times A \to Z$ be a homomorphism defined by $j(z_1,\cdots,z_m,a) = z_1$. Define, now, a homomorphism $g: H_1(Y;\mathbb{Z}) \to \mathbb{Z}$ by $g = j \circ i \circ p$. Then g is a non-trivial homomorphism. We'll show that $\mathrm{Hom}(f_*,\mathrm{id})(g) = 0$, which will contradict the fact that $\mathrm{Hom}(f_*,\mathrm{id})$ is a monomorphism. We have that

$$\mathrm{Hom}(f_*,\mathrm{id})(g) = g \circ f_*,$$

and hence, $\mathrm{Hom}(f_*,\mathrm{id})(g)(k) = g \circ f_*(k) = j \circ i \circ p \circ f_*(k) = j \circ i(0) = 0$, for every element k of $H_1(X;\mathbb{Z})$, since $f_*(k) \in \ker p$. This completes the proof of the theorem.

There exist simple examples which show that the converse of Theorem 3.2 is not true for every space X even if X is a Cantor n-manifold with $n \geq 3$, that is, a compact connected metric space such that no closed subset A with $\dim A \leq n-2$ separates X. The following is an example which shows that Problem 2 in [5] has a negative solution in the way it is stated.

3.3 *Example.* Let $Y = S^1 \times B^2$, where B^2 is the closed 2-ball and S^1 the 1-sphere, and let $X = Y/(\{t_0\} \times B^2)$ for some point $t_0 \in S^1$. It is clear that we can embed X into Y in such a way that the embedding $i: X \to Y$ induces an isomorphism $i_*: H_1(X;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$. Let $p: Y \to S^1$ be the first projection of Y onto S^1. We claim that X does not admit any weakly confluent mapping onto Y. Suppose, on the contrary, that $f: X \to Y$ is a weakly confluent mapping of X onto Y. By [5, Corollary 3.5], $f^*: H^1(Y;\mathbb{Z}) \to H^1(X;\mathbb{Z})$ is a monomorphism, and since $H^1(Y;\mathbb{Z}) \cong H^1(X;\mathbb{Z}) \cong \mathbb{Z}$, we infer that

f^* is an isomorphism. Let S_0 be a dyadic solenoid in Y, which is not contractible in Y. Since f is a weakly confluent mapping, there exists a compact, connected subset K of X such that $f(K) = S_0$. By [3, p. 542], $(p|S_0) \circ (f|K)$ non $\simeq 0$, which implies that

(1) $\qquad (p \circ f)|K$ non $\simeq 0$.

Let $\phi: Y \to X$ be the quotient mapping, and let $x_0 = \phi(\{t_0\} \times B^2)$. Then (1) implies that x_0 is a local cut-point of K, but it is not a cut-point of K. Consider the set $K \setminus \{x_0\}$ and let K' be a two-point compactification of $K \setminus \{x_0\}$. Let $h: K' \to K$ be the induced mapping. Then $h^{-1}(x_0) = \{a,b\}$ and $h|K' \setminus \{a,b\}$ is a homeomorphism of $K' \setminus \{a,b\}$ onto $K \setminus \{x_0\}$. It is clear, now, that $f \circ h$ is a mapping of K' onto S_0 such that $p \circ f \circ h \simeq 0$. This contradicts Fort's result in [3, p. 542], and completes the proof of the claim.

Let G be a group and H a subgroup of G. We say that G *has property (Tor) relative to* H provided that for every element g of G there exists a positive integer $n = n(g,H)$ such that $g^n \in H$. In case that H is a normal subgroup of G, this means that the group G/H is torsion. It is apparent also that if H has a finite index in G, then G has property (Tor) relative to H, but not conversely.

3.4 *Theorem. Let* $f: X \to Y$ *be a weakly confluent mapping from a compact connected* ANR X *onto an* ANR Y *such that every simple closed curve can be approximated by a spiral in* Y, *and such that* $f_{\#}\pi(X)$ *contains the commutator of* $\pi(Y)$. *Then* $\pi(Y)$ *has property (Tor) relative to* $f_{\#}\pi(X)$.

Proof. For simplicity, use the following notation: $\pi(X) = G$, $\pi(Y) = H$, $H_1(X;\mathbb{Z}) = G/G'$, and $H_1(Y;\mathbb{Z}) = H/H'$, where G' and H' are the commutators of G and H, respectively. Consider also the following diagram.

$$\begin{array}{ccccc} G & \xrightarrow{f_\#} & H & & \\ \downarrow g & & \downarrow h & & \\ G/G' & \xrightarrow{f_*} & H/H' & \xrightarrow{\Psi} & H_1(Y;\mathbb{Z})/f_*H_1(X;\mathbb{Z}) \end{array}$$

where g, h and Ψ are the quotient homomorphisms.

Let $\alpha \in H$. Then $h(\alpha) = \alpha H'$ and $\Psi[h(\alpha)] = \alpha H' f_*(G/G')$. By Theorem 3.2, $\Psi[h(\alpha)]$ is an element of finite order, and hence, there exists a positive integer n such that $\alpha^n H' \in f_*(G/G')$. Let $b \in G$ such that $f_*(bG') = \alpha^n H'$. Then $f_* g(b) = \alpha^n H'$. By the commutativity of the diagram (see [4, p. 51]) we have that $hf_\#(b) = \alpha^n H'$, which implies that $f_\#(b)H' = \alpha^n H'$ or $\alpha^{-n} f_\#(b) \in H'$. By the hypothesis $\alpha^{-n} f_\#(b) \in f_\#(G)$ which implies that $\alpha^{-n} \in f_\#(G)$, and since $f_\#(G)$ is a group, we also have that $\alpha^n \in f_\#(G)$, which completes the proof of the theorem.

3.5 *Corollary. Let* $f: M \to Y$ *be a mapping from a compact, connected* PL n*-manifold,* $n \geq 3$*, onto an* ANR Y *such that every simple closed curve can be approximated by a spiral in* Y*, and such that* $f_\#\pi(M)$ *contains the commutator of* $\pi(Y)$*. Then* f *is homotopic to an open mapping onto* Y *if and only if* f *is homotopic to a weakly confluent mapping.*

Proof. Since open mappings are weakly confluent, one direction is trivial. Suppose that f is homotopic to a weakly confluent mapping. Then by Theorem 3.4, $\pi(Y)$ has property (Tor) relative to $f_\#\pi(M)$. Since $f_\#\pi(M)$ contains

the commutator of $\pi(Y)$, by [9, p. 101], $f_{\#}\pi(M)$ is a normal subgroup of $\pi(Y)$, and hence, $\pi(Y)/f_{\#}\pi(M)$ is an abelian group, and since it is torsion and finitely presented, by [10, p. 49] it is a finite group. By Theorem 2.1, f is homotopic to an open mapping.

In Theorem 4.1 we will show, the condition in Corollary 3.5 that $f_{\#}\pi(X)$ contains the commutator of $\pi(Y)$ is necessary if no extra restrictions on Y are imposed. So the following questions can be posed.

Question 1. Let $f: X \to Y$ be a weakly confluent mapping from a compact connected PL n-manifold X onto a PL m-manifold Y, with $n, m \geq 3$. Is f homotopic to a light open mapping of X onto Y?

Question 2. Let $f: M \to Y$ be a mapping from a compact, connected, PL n-manifold, $n \geq 3$, into an ANR Y such that every simple closed curve can be approximated by a spiral in Y. If $\pi(Y)$ has property (Tor) relative to $f_{\#}\pi(M)$ is then f homotopic to a weakly confluent mapping of M onto Y?

Finally, we state the following--rather surprising--consequence of Corollary 3.5.

3.6 *Corollary. Let* $f: M \to Y$ *be a mapping from a compact, connected* PL n*-manifold* M, $n \geq 3$, *into a compact connected* ANR *such that every simple closed curve can be approximated by a spiral in* Y *and such that the fundamental group of* Y *is abelian. Then the following are equivalent:*

(1) f *is homotopic to a weakly confluent mapping of* M *onto* Y;

(2) f *is homotopic to an open mapping of* M *onto* Y;

(3) $\pi(Y)/f_{\#}\pi(X)$ *is a finite abelian group.*

4. Some constructions of weakly confluent mappings

In this section we construct weakly confluent mappings from any compact Hausdorff space of dimension at least three onto some compact ANRs with fundamental groups being the free products of finite groups, and we show that these are the weakly confluent images of the 3-cube, but they are not even the confluent images of any manifold with finite fundamental group.

4.1 *Theorem. Let* Y *be a compact, connected* ANR *which is the wedge of* ANRs *with finite fundamental groups. Then given any compact Hausdorff space* X *of dimension at least three, there exists a weakly confluent mapping of* X *onto* Y.

Proof. We shall first construct a weakly confluent mapping of the 3-cube B^3 onto Y. For this let Y be the wedge of the ANRs $Y_1, Y_2, \cdots, Y_m$ such that $Y_i \cap Y_j = \{y_0\}$ for every $i \neq j$; $i,j = 1,2,\cdots,m$, and such that for each i, $\pi(Y_i, y_0)$ is a finite group. Then

$$\pi(Y, y_0) = \pi(Y_1, y_0) * \pi(Y_2, y_0) * \cdots * \pi(Y_m, y_0),$$

that is, $\pi(Y, y_0)$ is the free product of $\pi(Y_1, y_0), \cdots, \pi(Y_m, y_0)$. Let also $B_1^3, B_2^3, \cdots, B_m^3$ be PL homeomorphic copies of B^3 with base points $x_1, x_2, \cdots, x_m$, respectively, being on the boundary of the cubes. Since $\pi(Y_i, y_0)$ is a finite

group, by Theorem 2.1 there exists an open and onto mapping

$$f_i': (B_i^3, x_i) \to (Y_i, y_0) \text{ for each } i = 1,2,\cdots,m.$$

Consider, now, for each $i = 1,2,\cdots,m$ a PL mapping $h_i: B_i^3 \to B_i^3$ defined as follows: Let D_i be a sufficiently small PL 2-disc centered at x_i on the boundary of B_i^3 and let $U_i(x_i)$ be a PL homeomorphic copy of $D_i x [0,1]$ such that $U_i(x_i) \cap \partial B_i^3 = D_i x \{0\}$. Then h_i maps $B_i^3 \setminus U_i(x_i)$ piecewise linearly and homeomorphically onto $B_i^3 \setminus \{x_i\}$ and $h_i[U_i(x_i)] = \{x_i\}$. Obviously, h_i is a monotone PL mapping of B_i^3 onto itself. Take $f_i = f_i' \circ h_i$. Since f_i' is open and h_i is monotone, f_i is a confluent mapping of (B_i^3, x_i) onto (Y_i, y_0) such that $f_i^{-1}(y_0) = U_i(x_i)$ for each $i = 1,2,\cdots,m$.

We, now, form a connected sum M of $B_1^3, B_2^3, \cdots, B_m^3$ as follows:

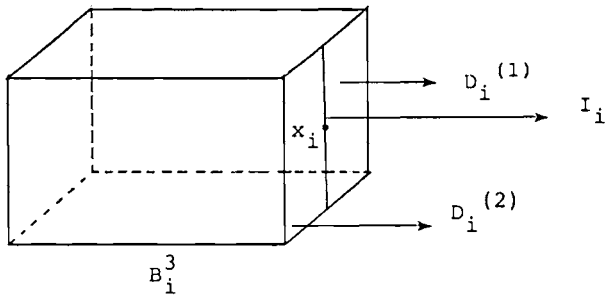

Figure 1

Consider D_i to consist of two PL 2-discs $D_i^{(1)}$ and $D_i^{(2)}$, which intersect on a PL arc I_i, which contains the point x_i (see Figure 1) and identify $D_1^{(1)}$ with $D_2^{(2)}$ such that

I_1 and I_2 are identified and also the points x_1 and x_2 are identified. Then identify $D_2^{(1)}$ with $D_3^{(2)}$ such that I_2 and I_3 are identified and also the points x_2 and x_3 are identified. Continue this process and finally identify $D_m^{(1)}$ with $D_1^{(2)}$ such that I_1 and I_m are identified. Call x_0 the resulting point from the identification of the points $x_1,\cdots,x_m$, and define a function $f: (M,x_0) \to (Y,y_0)$ by setting $f|B_i^3 = f_i$ for each $i = 1,2,\cdots,m$. Then by the construction f is a continuous and onto function, and M is a PL 3-cube. It remains to prove that f is a weakly confluent mapping. For this notice that if K is a subcontinuum of Y, then we have to consider the following cases:

(i) K does not contain y_0: Then there exists some i for which $K \subset Y_i \setminus \{y_0\}$, and hence, $f^{-1}(K) = f_1^{-1}(K)$. Since f_i is confluent, we infer that every component of $f^{-1}(K)$ maps onto K.

(ii) K contains y_0, but is not separated by y_0: Then we face this case as in case (i).

(iii) y_0 is a cut-point of K: Then $K \cap Y_i$ is always a subcontinuum of Y_i for each $i = 1,2,\cdots,m$, and hence, the component C_i of $f^{-1}(K \cap Y_i)$ which contains x_0 maps by f onto $K \cap Y_i$, since f_i is confluent and

$$f^{-1}(K \cap Y_i) = f_i^{-1}(K \cap Y_i) \cup f^{-1}(y_0).$$

Then $C = C_1 \cup C_2 \cup \cdots \cup C_m$ is a component of $f^{-1}(K)$ which is mapped by f onto K.

To complete, now, the proof of the theorem, let $g: X \to M$ be a weakly confluent mapping from X onto M. For the

existence of such a mapping see [7, Theorem 4.3]. Then $f \circ g: X \to Y$ is a weakly confluent mapping of X onto Y.

4.2 *Remarks.* (1) In Theorem 4.1, consider X to be any compact PL n-manifold, $n \geq 3$, with finite fundamental group, and Y to be the wedge of at least two ANRs with finite fundamental groups. Then there exist weakly confluent mappings of X onto Y, but there does not exist any confluent mapping from X onto Y, since any mapping $f: X \to Y$ is such that $f_{\#}\pi(X)$ has infinite index in $\pi(Y)$ (see [6]).

(2) A very simple consequence of Theorems 3.2 and 4.1 is that the abelianization of any free product of finitely many finite groups is a finite group.

The following problem is of interest:

Problem 3. Characterize all the weakly confluent images of the 3-cube.

References

[1] T. A. Chapman, *Lectures on Hilbert cube manifolds*, C.B.M.S. Regional Conference Series, Amer. Math. Soc. 28 (1977).

[2] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton University Press, Princeton, 1959.

[3] M. K. Fort, Jr., *Images of plane continua*, Amer. J. Math. 81 (1959), 541-546.

[4] M. J. Greenberg, *Lectures on algebraic topology*, W. A. Benjamin, Inc., Massachusetts, 1967.

[5] J. Grispolakis, *On Čech cohomology and weakly confluent mappings into* ANRs, Fund. Math. 111 (1981), 235-245.

[6] _____, *Confluent mappings and the fundamental group*, Bull. of Polish Acad. of Sci. 28 (1980), 643-646.

[7] _____ and E. D. Tymchatyn, *On confluent mappings and essential mappings*, Rocky Mountain J. Math. 11 (1981), 131-153.

[8] _____, *The fundamental group and weakly confluent mapping on* ANRs, *II*, Top. Proc. 11 (1986),

[9] A. G. Kurosh, *The theory of groups, Vol. I*, Chelsea Pub. Co., New York, 1955.

[10] S. Lang, *Algebra*, Addison-Wesley, Massachusetts, 1965.

[11] E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.

[12] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Rozprawy Matematyczne 50 (1968), 1-73.

[13] J. J. Walsh, *Monotone and open mappings on manifolds, I*, Trans. Amer. Math. Soc. 209 (1975), 419-432.

[14] _____, *Light open and open mappings on manifolds, II*, Trans. Amer. Math. Soc. 217 (1976), 271-284.

[15] _____, *Monotone and open mapping onto* ANRs, Proc. Amer. Math. Soc. 60 (1976), 286-288.

[16] J. West, *Mapping Hilbert cube manifolds to* ANRs: *A solution of a conjecture of Borsuk*, Annals of Math. 106 (1977), 1-18.

[17] G. T. Whyburn, *Analytic topology*, Colloq. Publ. Amer. Math. Soc. 28 (1942).

Technical University of Crete

Chania 73100

Crete, Greece