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COMPACT GROUPS AND ABSOLUTE EXTENSORS

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Abstract

We discuss compact Hausdorff groups from the
point of view of the general theory of absolute ex-
tensors. In particular, we characterize the class
of simple, connected and simply connected com-
pact Lie groups as $AE(2)$-groups the third ho-
motopy group of which is $\mathbb{Z}$. This is the converse
of the corresponding result of R. Bott.

The following result [3, Theorem 6.57 and Corollary 6.59]
plays an important role in the theory of compact groups.

Theorem A (Structure Theorem for Compact Groups).
*Let G be a connected compact group. Then there exist a con-
tinuous homomorphism*

$$p\colon Z_0(G) \times \prod\{L_t\colon t \in T\} \to G$$

*where $Z_0(G)$ stands for the identity component of the center of
G and L_t is a simple, connected and simply connected compact
Lie group, $t \in T$, such that $\ker(p)$ is isomorphic to a zero-
dimensional central subgroup of G.*

*If G itself is a compact connected n-dimensional Lie group,
then $\ker(p)$ is finite, the indexing set T is also finite and $Z_0(G)$
is a torus of codimension $|T|$ (i.e. $Z_0(G)$ is isomorphic to the
product of $n - |T|$ copies of the circle group $\mathbb{T}$).*

In many instances the above statement allows us to split
a reasonably stated problem about compact connected groups
into the abelian and nonabelian parts (explicitly indicated in
the domain of p). Obviously its efficiency depends on our abil-
ity to recognize these parts separately. This is why the second
part of Theorem A is much more precise than the first. One
aspect of this general point of view is reflected in the following
two questions:

(I) What is a (topological) characterization of tori, i.e. of
   (possibly uncountable) powers $\mathbb{T}^\tau$ of the circle group $\mathbb{T}$.

(II) What is a (topological) characterization of simple, con-
   nected and simply connected compact groups?

Question (I) has recently been answered [1, Theorem E]
within the general theory of absolute extensors (consult [2] for
a comprehensive discussion of related topics):

Theorem B. *The following conditions are equivalent for a
compact abelian group G:*

- *G is a torus.*

- *G is an $AE(1)$-compactum (= compact absolute extensor
  in dimension 1).*

In these notes we give a complete solution of question (II).
Characterization, as in the abelian case [1], is given within the
theory of $AE(n)$-spaces.

Lie groups, topologically being manifolds, are locally n-con-
nected for each $n \geq 0$ (even locally contractible). Two basic
examples of global connectivity properties are, of course, con-
nectedness and simple connectedness. Let us examine these
concepts more closely. Connected manifolds (because of their
local niceness) are arcwise connected (i.e. 0-connected). Note
that this is not true for non locally connected groups. Unlike

connectedness, arcwise connectedness is a concept of a homo-
topy theoretical nature and fits into a general extension theory.
Indeed, consider the following extension problem $E(Z, Z_0, g_0)$
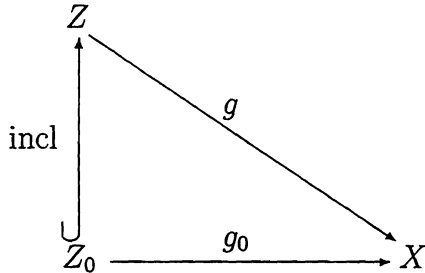
which asks whether a map g_0, defined on a closed subspace Z_0
of a compactum Z, with values in X, has an extension g over
the whole Z. Clearly a compactum X is arcwise connected
precisely when the extension problem $E(B^1, \partial B^1, g_0)$, where
B^1 is the 1-dimensional disk, is solvable for any g_0. Simi-
larly the class of compacta for which the extension problem
$E(B^2, \partial B^2, g_0)$, where B^2 is the 2-dimensional disk, is solvable
for any g_0 coincides with the class of simply connected spaces.
An important observation (see for instance [4, Chapter 7, §53,
Section IV, Theorems 1 and 1′]) here is that for a *metrizable*
locally 0-connected compactum X the 0-connectedness of X
guarantees the solvability of the extension problem $E(Z, Z_0, g_0)$
for any choice of at most 1-dimensional compactum Z, closed
subspace Z_0 of Z, and map g_0. In other words, X in this
case is an absolute extensor in dimension 1 (shortly, $AE(1)$-
compactum). This fact sometimes is expressed by writing
$LC^0 \cap C^0 = AE(1)$. These concepts can obviously be defined
for any n and as is well known the strictly decreasing sequences

$$AE(1) \supset AE(2) \supset \cdots \supset AE(n) \supset AE(n+1) \supset \cdots$$

and

$$LC^0 \cap C^0 \supset LC^1 \cap C^1 \supset \cdots \cdots \supset LC^{n-1} \cap C^{n-1}$$

$$\supset LC^n \cap C^n \supset \cdots$$

are identical (i.e., $AE(n) = LC^{n-1} \cap C^{n-1}$) in the presence of metrizability. Nevertheless these concepts differ (i.e. $AE(n) \subsetneq LC^{n-1} \cap C^{n-1}$) in general. For compact groups the difference between these concepts is very delicate. To see this it suffices to compare Theorem B with the result from [7]: the validity of the statement "an arcwise connected (i.e. 0-connected) compact abelian group is a torus" is undecidable within ZFC.

Thus every connected Lie group topologically is an $AE(1)$-space and every connected and simply connected Lie group is an $AE(2)$-space.

Theorem C. *The following conditions are equivalent for a compact group G:*

(a) G is a simply connected $AE(1)$-compactum.

(b) G is an $AE(2)$-compactum.

(c) G is an $AE(3)$-compactum.

(d) G is a product of simple, connected and simply connected compact Lie groups.

Proof. Implications (c) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (a) are trivial.

(a) $\Longrightarrow$ (d). It follows from the proof of Theorem A [3] that there exists a continuous homomorphism $\alpha \colon G' \times Z_0(G) \to G$ where G' is the commutator subgroup of G, $Z_0(G)$ is the connected component of the center of G, and $\ker(\alpha)$ is a zero-dimensional compact group isomorphic to the (closed and central) subgroup $G' \cap Z_0(G)$ of G. Further, there exists a continuous homomorphism $\beta \colon \prod\{L_t \colon t \in T\} \to G'$ where each L_t, $t \in T$, is a simple, connected and simply connected compact Lie group, and $\ker(\beta)$ is zero-dimensional. Obviously, the kernel of the homomorphism p, defined as the composition

$$p = \alpha \circ (\beta \times \mathrm{id}_{Z_0(G)}) \colon \prod\{L_t \colon t \in T\} \times Z_0(G) \xrightarrow{\beta \times \mathrm{id}}$$
$$G' \times Z_0(G) \xrightarrow{\alpha} G,$$

is zero-dimensional. Let us show that $|\ker(p)| = 1$. Indeed, assuming that $|\ker(p)| > 1$ and remembering that $\dim \ker(p) = 0$, we can find a compact group $\tilde{G}$ and two continuous homomorphisms

$$p_1 \colon \prod\{L_t \colon t \in T\} \times Z_0(G) \to \tilde{G} \quad \text{and} \quad p_2 \colon \tilde{G} \to G$$

such that $p = p_2 p_1$, $\ker(p_2)$ is finite and $|\ker(p_2)| > 1$. Clearly p_2 is a bundle (since $\ker(p_2)$ is a finite group) and G, as an $AE(1)$-compactum, is arcwise connected and locally arcwise connected. Also, by (a), $\pi_1(G) = 0$. Consequently, by [9, Corollary on p.66], p_2 is a trivial bundle, i.e. $\tilde{G} \cong G \times \ker(p_2)$. Now observe that $\tilde{G}$, as a continuous image of $\prod\{L_t \colon t \in T\} \times Z_0(G)$, is connected. This implies that $\ker(p_2)$ is a singleton which is impossible. This contradiction shows that $|\ker(p)| = 1$ and hence $\prod\{L_t \colon t \in T\} \times Z_0(G) \cong G$. From this we conclude that $Z_0(G)$, as a retract of G, is an $AE(1)$-compactum with $\pi_1(Z_0(G)) = 0$. An abelian group which is an $AE(1)$-compactum is a torus (Theorem B). But the only simply connected torus is the singleton. Thus $|Z_0(G)| = 1$ and $G \cong \prod\{L_t \colon t \in T\}$.

(d) $\implies$ (c). Suppose that G is a product $\prod\{L_t \colon t \in T\}$ of simple, connected and simply connected compact Lie groups. By Theorem of H. Cartan [5, Theorem 3.7], $\pi_2(L_t) = 0$, $t \in T$. By the local contractibility of L_t, this means that L_t is an $AE(3)$-compactum, $t \in T$. Then G, as a product of $AE(3)$-compacta, is an $AE(3)$-compactum. The proof is complete. $\square$

Remark 1. Implication (a) $\implies$ (b) of Theorem C extends a result of H. Cartan which states that if G is a connected and (semi-) simple compact Lie group, then $\pi_2(G) = 0$.

We now present corollaries of Theorem C. The following answers question (II).

Corollary 1. *The following conditions are equivalent for a non-trivial compact group G:*

(i) G is an $AE(2)$-group with $\pi_3(G) = \mathbb{Z}$.

(ii) G is a simple, connected and simply connected Lie group.

Proof. (i) $\Longrightarrow$ (ii). By Theorem C, G is isomorphic to the product $\prod\{L_t: t \in T\}$ of simple, connected and simply connected compact Lie groups. Let $T' = \{t \in T: |L_t| > 1\}$. By (i) and a result of R. Bott [5, Theorem 3.8], we then have

$$\mathbb{Z} = \pi_3(G) = \pi_3\left(\prod\{L_t: t \in T'\}\right) = \prod\{\pi_3(L_t): t \in T'\} = \mathbb{Z}^{T'}.$$

This obviously is possible precisely when $|T'| = 1$ which implies that G is isomorphic to some L_t and hence is a simple, connected and simply connected compact Lie group.

(ii) $\Longrightarrow$ (i). Every connected and simply connected Lie group is an $AE(2)$-space. Since G is simply connected, it is non abelian. Since G is simple, by the above cited result of R. Bott, $\pi_3(G) = \mathbb{Z}$.   □

Remark 2. Implication (ii) $\Longrightarrow$ (i) of Corollary 1 is due to R. Bott.

Corollary 2. *There is no non-trivial compact $AE(4)$-group.*

Proof. Let G be a compact $AE(4)$-group. Since every $AE(4)$-compactum is an $AE(3)$-compactum, it follows from Theorem C, that G is a product $\prod\{L_t: t \in T\}$ of simple, connected and simply connected compact Lie groups. Since each retract of an $AE(4)$-compactum is an $AE(4)$-compactum, it follows that L_t is an $AE(4)$-compactum, $t \in T$. Then $\pi_3(L_t) = 0$ for each $t \in T$. This contradicts Corollary 1 and completes the proof.   □

The concept of an $ANE(4)$-space (=absolute *neighborhood* extensor in dimension 4) is obtained by requesting (in the extension problem $E(Z, Z_0, g_0)$ with any at most 4-dimensional

space Z, any closed subspace Z_0 of Z, and any continuous g_0)
the existence of an extension g (of g_0) defined not on the whole
Z (as we did while defining the concept of $AE(4)$) but only on
a neighborhood of Z_0 in Z. Clearly every $ANE(4)$-space is
locally 3-connected but not vice versa. It is well known that a
locally compact group is a Lie group if and only if it is finite-
dimensional and locally connected [6]. It is easy to produce
examples of non-metrizable locally connected compact groups,
whereas a finite dimensional connected locally compact group
is always metrizable [8]. Roughly speaking our next Corollary
shows what degree of local connectedness forces an infinite di-
mensional compact group to be metrizable.

Corollary 3. *Every locally compact $ANE(4)$-group is metriz-
able.*

Proof. Let G be a locally compact $ANE(4)$-group. By [8, The-
orem 16′], G is homeomorphic to the product $\mathbb{Z} \times (\mathbb{Z}_2)^\tau \times \mathbb{R}^n \times L$
for some integer n, a cardinal number τ and a compact group
L. Since each retract of an $ANE(4)$-space is an $ANE(4)$-space,
we conclude that τ is finite. Therefore it suffices to show that
L is metrizable.

By [2, Lemma 8.2.1], L is isomorphic to the limit of an ω-
spectrum $\mathcal{S}_L = \{L_\alpha, p_\alpha^\beta, A\}$ consisting of metrizable compact
groups L_α and continuous homomorphisms $p_\alpha^\beta \colon L_\beta \to L_\alpha$. Ob-
serve that L, as a retract of G, is an $ANE(4)$-compactum.
Consequently, by [2, Theorems 1.3.4 and 6.3.2], there exists a
cofinal (in particular, non-empty) subset B of the indexing set
A such that the limit projection $p_\alpha \colon L \to L_\alpha$ is 4-soft for each
$\alpha \in B$. Then, by [2, Corollary 6.1.17 and Proposition 6.1.21],
the fiber $\ker(p_\alpha)$ is a compact $AE(4)$-group. By Corollary 2,
$|\ker(p_\alpha)| = 1$. Therefore p_α is an isomorphism. Since L_α is
metrizable, so is L.   □

References

[1] A. Chigogidze, *Topological characterization of torus groups*, Topology Appl., to appear.

[2] —, *Inverse Spectra*, North Holland, Amsterdam, 1996.

[3] K. H. Hofmann, *Introduction to the theory of compact groups, Part II*, Tulane University lecture notes, Tulane University, 1968.

[4] K. Kuratowski, *Topology*, vol. II, Academic Press, New York, 1968.

[5] M. Mimura, *Homotopy theory of Lie groups*, in: Handbook of Algebraic Topology, North Holland, Amsterdam, 1995, 951–991.

[6] L. S. Pontryagin, *Topological Groups*, Princeton University Press, Princeton, New Jersey, 1946.

[7] S. Shelah, *Infinite abelian groups, Whitehead problem and some constructions*, Israel J. Math., **18** (1974), 243–256.

[8] E. G. Skljarenko, *On the topological structure of locally bicompact groups and their quotient spaces*, Mat. Sbornik, **60** (1963), 57–82.

[9] N. Steenrod, *The topology of fibre bundles*, Princeton University Press, Princeton, New Jersey, 1951.

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